

Existence of blueshifts in quasi-spherical Szekeres spacetimes

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In Lemaître – Tolman (L–T) models, light rays emitted *radially* at the Big Bang (BB) at such radial coordinates r where the bang-time function $t_B(r)$ has $dt_B/dr \neq 0$ reach every observer with *infinite blueshift*, $z = -1$. Consequently, there exist rays, emitted soon after the BB, that will reach later observers with finite blueshift ($-1 < z < 0$). But in spacetimes without symmetry there are no radial directions. The question thus arises whether blueshifts can exist at all in the Szekeres models that contain L–T as a limit, but in general have no symmetry. The aim of the present paper is to show that strong blueshifts can be generated in quasi-spherical Szekeres (QSS) models. It is shown that in an axially symmetric QSS model, infinite blueshift can appear only on axial rays, which intersect every space orthogonal to the dust flow on the symmetry axis. In an exemplary QSS model it is numerically shown that if such a ray is emitted from the Big Bang where $dt_B/dr \neq 0$, then indeed observers see it with $z \approx -1$. Rays emitted shortly after the BB and running close to the symmetry axis will reach the observers with a strong blueshift, too. Then, in a toy QSS model that has no symmetry, it was shown by numerical calculations that two null lines exist such that rays in their vicinity have redshift profiles similar to those in a vicinity of the axial rays in the axially symmetric case. This indicates that rays generating infinite blueshifts exist in general QSS spacetimes and are concentrated around two directions.

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I. MOTIVATION AND BACKGROUND

In the spherically symmetric Lemaître [1] – Tolman [2] (L–T) cosmological models some of the radial null geodesics have a peculiar property. Suppose the point P_o on the radial null geodesic G lies later than the Big Bang (BB), and we follow G to the past until it intersects the BB at P_e . The redshift z observed at P_o depends on the slope of the BB at P_e . Namely, if the bang-time function $t_B(r)$ has $dt_B/dr \neq 0$ at P_e , then $z(P_o) = -1$. On all other geodesics (G being nonradial *or* $dt_B/dr = 0$ at P_e), $z(P_o) = \infty$ [3–5]. The $z < 0$ property, referred to as blueshift, means that the observed frequency ν_o of an electromagnetic wave is greater than the emitted frequency ν_e , and $z \rightarrow -1$ implies $\nu_o \rightarrow \infty$. Since the L–T model is unrealistic before the last-scattering hypersurface (LSH), in the real Universe $z(P_o)$ would always be greater than -1 , and ν_o would always be finite.

In L–T models the meaning of a radial or nonradial direction is obvious. Not so in Szekeres models [6–8], which in general have no symmetry. But the L–T models are contained in those of Szekeres as a spherically symmetric limit. So, it is an interesting question whether $z < 0$ can arise also in Szekeres models, and on which rays. This is the subject of the present paper. Only the quasi-spherical Szekeres (QSS) models are considered here because the quasiplane and quasihyperbolic models are still poorly understood [9–11]. It is shown that the extra flexibility

provided by the dipole component of mass density in the Szekeres models makes it easier to get strong blueshifts along certain directions while the rays that generate those blueshifts are fewer. Numerical experiments showed that such rays do exist in the axially symmetric case; they intersect each space of constant t on the symmetry axis.

The models considered here are not related to the actual cosmological observations; they are meant to illustrate the improvements in generating blueshifts achieved in comparison with the L–T models.

In Sec. II, properties of the QSS models are briefly described. In Sec. III, the notion of an origin is defined and the behaviour of the arbitrary functions of the QSS model at the origin is discussed. In Sec. IV, several useful formulae are listed for reference. In Sec. V, the equations of null geodesics in a general QSS spacetime are introduced, and basic properties of those geodesics are briefly discussed. In Sec. VI, the behaviour of redshift along null geodesics is discussed, and necessary conditions for infinite blueshifts are derived.

In Sec. VII, equations of null geodesics in an axially symmetric QSS are displayed. In Sec. VIII, the equation of an extremum redshift surface is derived for those null geodesics that proceed along the symmetry axis in an axially symmetric QSS (they will be called axial). In Sec. IX, an example of an axially symmetric QSS metric is introduced, and properties of axial null geodesics in it are discussed in Sec. X. These geodesics do display strong blueshifts (SB) (i.e. $z \approx -1$ on them) if they originate at the BB at points where $dt_B/dr \neq 0$. In Sec. XI, such null geodesics in the same QSS metric are discussed that run close to the symmetry axis. They can generate

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strong blueshifts when they originate close to the BB. (When they originate exactly at the BB, all observers see infinite redshifts, independently of dt_B/dr .)

In Sec. XII, an exemplary QSS model is introduced that has no symmetry. By numerical calculations it is shown that such rays exist in this model, on which the redshift profiles have the same shape as on the rays running close to the symmetry axis in the axially symmetric model. In particular, light sources lying on those rays close to, but later than the BB, would be observed with finite blueshifts. This is an indication that rays generating SB from the BB should exist in their vicinity. However, in all known cases rays with SB are unstable: an arbitrarily small perturbation of the observation point or direction changes SB into infinite redshift. Therefore, for tracking such rays numerically we must know how to keep them exactly on their unstable paths (as is the case with radial rays in the L-T models and with axial rays in the axially symmetric QSS models).

Section XIII contains a summary of the results. Details of some calculations are explained in the appendices.

II. THE QUASISPHERICAL SZEKERES SPACETIMES

The metric of the quasispherical Szekeres spacetimes [12] can be written as (notation adapted to [8])

$$ds^2 = dt^2 - \frac{(\Phi_{,r} - \Phi \mathcal{E}_{,r}/\mathcal{E})^2}{1 + 2E(r)} dr^2 - \left(\frac{\Phi}{\mathcal{E}}\right)^2 (dx^2 + dy^2), \quad (2.1)$$

where

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{S}{2} \left[\left(\frac{x-P}{S}\right)^2 + \left(\frac{y-Q}{S}\right)^2 + 1 \right], \quad (2.2)$$

$P(r)$, $Q(r)$, $S(r)$ and $E(r)$ being arbitrary functions such that $S \neq 0$ and $E \geq -1/2$ at all r ($E = -1/2$ can occur at isolated values of r , but not on open intervals [13]).

The source in the Einstein equations is dust ($p = 0$), and the coordinates of (2.1) are comoving, so the velocity field of the dust is $u^\alpha = \delta_0^\alpha$. The surfaces of constant t and r are nonconcentric spheres, and (x, y) are the stereographic coordinates on each sphere. At a fixed r , the relation between (x, y) and the spherical coordinates is

$$\begin{aligned} x &= P + S \tan(\vartheta/2) \cos \varphi, \\ y &= Q + S \tan(\vartheta/2) \sin \varphi. \end{aligned} \quad (2.3)$$

The functions (P, Q, S) determine the positions of the centres of the spheres in the spaces of constant t (see an example in Sec. XII). The function $\Phi(t, r)$ is determined by the same evolution equation as in the L-T models:

$$\Phi_{,t}^2 = 2E(r) + \frac{2M(r)}{\Phi} - \frac{1}{3}\Lambda\Phi^2, \quad (2.4)$$

where Λ is the cosmological constant and $M(r)$ is an arbitrary function. Any solution of (2.4) depends on t

through the combination $(t - t_B(r))$, where $t_B(r)$ is still one more arbitrary function; $t = t_B(r)$ is the BB time, at which $\Phi(t_B, r) = 0$.

In the following, we shall assume $\Phi_{,t} > 0$ (the Universe is expanding) and $\Lambda = 0$. The solutions of (2.4) under these assumptions are presented in Appendix A; they are the same as the Friedmann solutions [8].

The mass density implied by (2.1) is

$$\kappa\rho = \frac{2(M_{,r} - 3M\mathcal{E}_{,r}/\mathcal{E})}{\Phi^2(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E})}, \quad \kappa \stackrel{\text{def}}{=} \frac{8\pi G}{c^2}. \quad (2.5)$$

In choosing the arbitrary functions, one must take care that the resulting mass density in the region being considered is positive and finite. The conditions that ensure this were worked out in Ref. [13], and they are:

$$\frac{M_{,r}}{3M} \geq \frac{\sqrt{(S_{,r})^2 + (P_{,r})^2 + (Q_{,r})^2}}{S} \quad \forall r, \quad (2.6)$$

$$\frac{E_{,r}}{2E} > \frac{\sqrt{(S_{,r})^2 + (P_{,r})^2 + (Q_{,r})^2}}{S} \quad \forall r. \quad (2.7)$$

These inequalities ensure that [13]

$$\frac{M_{,r}}{3M} \geq \frac{\mathcal{E}_{,r}}{\mathcal{E}}, \quad \frac{E_{,r}}{2E} > \frac{\mathcal{E}_{,r}}{\mathcal{E}} \quad \forall r. \quad (2.8)$$

As first noted by Szekeres [7] and elaborated by de Souza [14], the density distribution (2.5) is that of a mass-dipole superposed on a spherically symmetric monopole. The dipole contribution is generated by the term $\mathcal{E}_{,r}/\mathcal{E}$ and vanishes on the set where $\mathcal{E}_{,r} = 0$. The extrema of density coincide with the extrema of $\mathcal{E}_{,r}/\mathcal{E}$: the density is minimum where $\mathcal{E}_{,r}/\mathcal{E}$ is maximum and vice versa [13].

III. THE ORIGIN AND THE BEHAVIOUR OF THE ARBITRARY FUNCTIONS AT IT

It is not necessary for a Szekeres spacetime to have an origin. Spacetimes without an origin have cylindrical topology of the constant-time subspaces; in the spherically symmetric limit they do not contain the center of symmetry. This configuration is somewhat exotic, so we shall assume that an origin exists. It is the set at which each sphere of constant t and r in (2.1) has zero radius, i.e. where $\Phi = 0$ at all $t > t_B$. Multiplying (2.4) by Φ , anticipating that $|\Phi_{,t}|_{\text{origin}} < \infty$, and then taking the result at the origin we obtain that

$$M_{\text{origin}} = 0. \quad (3.1)$$

Since M depends only on r , the origin worldline is a line of constant r , i.e. the origin is comoving and coincides always with the same dust particle.

The metric (2.1) is covariant with the transformations $r = f(r')$, where $f(r')$ is an arbitrary function. These can be used to give one of the arbitrary functions a convenient shape. It is advantageous to choose r so that

$$M(r) = M_0 r^3, \quad (3.2)$$

where M_0 is a constant, which can be given an arbitrary nonzero value by a further transformation $r' = Ar''$, where A is a constant. We shall assume (3.2) and $M_0 = 1$, but M_0 will be kept in all formulae to avoid confusion about dimensions of various derived quantities.

Equations (3.1) and (3.2) imply

$$r_{\text{origin}} = 0. \quad (3.3)$$

We assume¹

$$S > 0 \text{ everywhere} \implies \mathcal{E} > 0 \text{ everywhere}. \quad (3.4)$$

Since at $r = 0$ both $M = 0$ and $\Phi = 0$, using (3.4) one can calculate the limit

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\Phi^3}{M} &\equiv \lim_{r \rightarrow 0} \frac{(\Phi/\mathcal{E})^3}{M/\mathcal{E}^3} = \lim_{r \rightarrow 0} \frac{3\Phi^2(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E})}{M_{,r} - 3M\mathcal{E}_{,r}/\mathcal{E}} \\ &\equiv \frac{6}{\kappa\rho(t, r)} \Big|_{r=0} \stackrel{\text{def}}{=} \frac{R^3(t)}{M_0}. \end{aligned} \quad (3.5)$$

This limit is thus finite at those points of the origin where $\rho(t, 0) \neq 0$. At these points, Φ must have the form

$$\Phi = r[R(t) + \mathcal{B}(t, r)], \quad (3.6)$$

where \mathcal{B} has the property

$$\lim_{r \rightarrow 0} \mathcal{B} = 0. \quad (3.7)$$

Note that no approximation is involved in (3.6); this is a reparametrisation of Φ that respects (3.5). In the Friedmann limit $\mathcal{B} \equiv 0$, and R becomes the scale factor.

Using (3.6) and (3.2) in (2.4) we conclude that E must have the form

$$2E = r^2(-k + \mathcal{F}(r)), \quad (3.8)$$

where k is a constant and $\mathcal{F}(r)$ has the property

$$\lim_{r \rightarrow 0} \mathcal{F} = 0. \quad (3.9)$$

Equation (3.8) is also an exact formula. In the Friedmann limit $\mathcal{F} \equiv 0$, and k becomes the curvature index.

IV. SOME USEFUL FORMULAE

In the next sections we shall need to know the behaviour of $\Phi_{,r}$ and $\Phi_{,tr}$ at the BB and at $r \rightarrow 0$. We have ([5], Eqs. (3.9) and (3.10)):

$$\begin{aligned} \Phi_{,r} &= \left(\frac{M_{,r}}{M} - \frac{E_{,r}}{E} \right) \Phi \\ &+ \left[\left(\frac{3E_{,r}}{2E} - \frac{M_{,r}}{M} \right) (t - t_B) - t_{B,r} \right] \Phi_{,t}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \Phi_{,tr} &= \frac{E_{,r}}{2E} \Phi_{,t} \\ &- \frac{M}{\Phi^2} \left[\left(\frac{3E_{,r}}{2E} - \frac{M_{,r}}{M} \right) (t - t_B) - t_{B,r} \right]. \end{aligned} \quad (4.2)$$

A. Limits at the BB

These limits are calculated at $r > 0$. Limits at the points where simultaneously $t = t_B$ and $r = 0$ are nonunique: they depend on the detailed shapes of the arbitrary functions and on the path of approach to such a point; see an example for the L-T model in Ref. [8].

With $t \rightarrow t_B$ (so $\Phi \rightarrow 0$) we find using (2.4):

$$\lim_{t \rightarrow t_B} \Phi_{,t} = \lim_{t \rightarrow t_B} \Phi_{,tr} = \infty, \quad (4.3)$$

$$\lim_{t \rightarrow t_B} [(t - t_B)/\Phi] = 0, \quad (4.4)$$

$$\lim_{t \rightarrow t_B} [(t - t_B)\Phi_{,t}] = \lim_{t \rightarrow t_B} (\Phi\Phi_{,t}) = 0, \quad (4.5)$$

$$\lim_{t \rightarrow t_B} [(t - t_B)\Phi_{,t}/\Phi] = \frac{2}{3}. \quad (4.6)$$

From (4.3) and (4.5) we find

$$\lim_{t \rightarrow t_B} \Phi_{,r} = -t_{B,r} \lim_{t \rightarrow t_B} \Phi_{,t}. \quad (4.7)$$

Thus, on a curve that hits the BB where $dt_B/dr \neq 0$ we have $\lim_{t \rightarrow t_B} \Phi_{,r} = \pm\infty$; the sign in front of ∞ is the sign of $(-dt_B/dr)$. On a curve that hits the BB where $dt_B/dr = 0$ we have $\lim_{t \rightarrow t_B} \Phi_{,r} = 0$.

Further, it follows from (4.6) that

$$\lim_{t \rightarrow t_B} \left(\frac{\Phi_{,r}}{\Phi} \right) = \frac{M_{,r}}{3M} - t_{B,r} \lim_{t \rightarrow t_B} \left(\frac{\Phi_{,t}}{\Phi} \right). \quad (4.8)$$

This limit is finite when $dt_B/dr = 0$ at the intersection of the path of approach with the BB, and is infinite otherwise; the sign of the infinity is again the sign of $(-t_{B,r})$.

B. Limits at the origin

The limits given below are calculated at $t > t_B$; the reason for avoiding the point $(r, t) = (0, t_B)$ is the same as in the previous subsection.

Using (2.4), (3.2) and (3.6) – (3.8) we find:

$$\lim_{r \rightarrow 0} \Phi_{,t} = 0, \quad (4.9)$$

$$\lim_{r \rightarrow 0} \frac{\Phi_{,t}}{\Phi} = \frac{R_{,t}}{R} < \infty, \quad (4.10)$$

$$\lim_{r \rightarrow 0} \Phi_{,r} = R < \infty, \quad (4.11)$$

$$\lim_{r \rightarrow 0} \Phi_{,r}/\Phi = \infty, \quad (4.12)$$

¹ At points where $S = 0$ we have $\Phi/\mathcal{E} = 0$ in (2.1), so they are additional origins. The Riemann tensor will be finite at those points if $|\mathcal{E}_{,r}| < \infty$ there, see Appendix B. The metric does not change under the substitution $S = -S$, so the assumption $S > 0$ is not a limitation.

$$\lim_{r \rightarrow 0} \Phi_{,tr} = R_{,t} < \infty. \quad (4.13)$$

V. NULL GEODESICS IN QUASISPHERICAL SZEKERES SPACETIMES

We denote

$$(k^t, k^r, k^x, k^y) \stackrel{\text{def}}{=} \frac{d(t, r, x, y)}{d\lambda}, \quad (5.1)$$

where λ is the affine parameter, and

$$\mathcal{N} \stackrel{\text{def}}{=} \Phi_{,r} - \Phi \mathcal{E}_{,r} / \mathcal{E}. \quad (5.2)$$

Then the equations of geodesics for (2.1) are [15]

$$\frac{dk^t}{d\lambda} + \frac{\mathcal{N}\mathcal{N}_{,t}}{1+2E} (k^r)^2 + \frac{\Phi\Phi_{,t}}{\mathcal{E}^2} [(k^x)^2 + (k^y)^2] = 0, \quad (5.3)$$

$$\begin{aligned} \frac{dk^r}{d\lambda} + 2\frac{\mathcal{N}_{,t}}{\mathcal{N}} k^t k^r + \left(\frac{\mathcal{N}_{,r}}{\mathcal{N}} - \frac{E_{,r}}{1+2E} \right) (k^r)^2 \\ - 2\Phi \frac{(\mathcal{E}_{,r}/\mathcal{E})_{,x} k^x + (\mathcal{E}_{,r}/\mathcal{E})_{,y} k^y}{\mathcal{N}} k^r \\ - \frac{\Phi}{\mathcal{E}^2} \frac{1+2E}{\mathcal{N}} [(k^x)^2 + (k^y)^2] = 0, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \frac{dk^x}{d\lambda} + 2\frac{\Phi_{,t}}{\Phi} k^t k^x + \frac{\mathcal{E}^2 \mathcal{N}}{\Phi(1+2E)} \left(\frac{\mathcal{E}_{,r}}{\mathcal{E}} \right)_{,x} (k^r)^2 \\ + 2\frac{\mathcal{N}}{\Phi} k^r k^x - \frac{\mathcal{E}_{,x}}{\mathcal{E}} (k^x)^2 \\ - 2\frac{\mathcal{E}_{,y}}{\mathcal{E}} k^x k^y + \frac{\mathcal{E}_{,x}}{\mathcal{E}} (k^y)^2 = 0, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \frac{dk^y}{d\lambda} + 2\frac{\Phi_{,t}}{\Phi} k^t k^y + \frac{\mathcal{E}^2 \mathcal{N}}{\Phi(1+2E)} \left(\frac{\mathcal{E}_{,r}}{\mathcal{E}} \right)_{,y} (k^r)^2 \\ + 2\frac{\mathcal{N}}{\Phi} k^r k^y + \frac{\mathcal{E}_{,y}}{\mathcal{E}} (k^x)^2 \\ - 2\frac{\mathcal{E}_{,x}}{\mathcal{E}} k^x k^y - \frac{\mathcal{E}_{,y}}{\mathcal{E}} (k^y)^2 = 0. \end{aligned} \quad (5.6)$$

The geodesics determined by (5.3) – (5.6) are null when

$$(k^t)^2 - \frac{\mathcal{N}^2 (k^r)^2}{1+2E(r)} - \left(\frac{\Phi}{\mathcal{E}} \right)^2 [(k^x)^2 + (k^y)^2] = 0. \quad (5.7)$$

Note that $k^r \neq 0$ over any open interval of a null geodesic: otherwise, $dk^r/d\lambda = 0$ in that interval, and (5.4) would imply $k^x = k^y = 0$; such a geodesic would be timelike. However, $k^r = 0$ is allowed at isolated points. Thus, r can be used as a parameter on any arc of a null geodesic, on which k^r does not change sign.

Let the subscript o refer to the observation point. We will mostly consider past-directed rays, on which $k^t < 0$. Since the affine parameter along each single geodesic is determined up to the transformations $\lambda = a\lambda' + b$, where a and b are constants, it can be chosen such that

$$k_o^t = -1, \quad (5.8)$$

and this choice will be made throughout this paper. Then, from (5.7) we have

$$(k_o^x)^2 + (k_o^y)^2 \leq \left(\frac{\mathcal{E}_o}{\Phi_o} \right)^2; \quad (5.9)$$

the equality occurs only when $k_o^r = 0$, i.e. when the null geodesic passes through the observation event tangentially to the hypersurface of constant r .

The coefficient \mathcal{N}/Φ in front of $(k^r)^2$ and of k^r in (5.5) and (5.6) becomes infinite at the origin, as follows from (4.12). Thus, when running a numerical calculation of a geodesic through the origin the limits of $(\mathcal{E}_{,r}/\mathcal{E})_{,x} (k^r)^2/\Phi$, $(\mathcal{E}_{,r}/\mathcal{E})_{,y} (k^r)^2/\Phi$, $k^r k^x/\Phi$ and $k^r k^y/\Phi$ have to be evaluated exactly. If they are finite, then it is best to choose the origin as the initial point. See Sec. XI for an example.

VI. REDSHIFT IN THE QUASISPHERICAL SZEKERES SPACETIMES

The general formula for redshift along a ray emitted at P_e and observed at P_o is [16]

$$1+z = \frac{(u_\alpha k^\alpha)_e}{(u_\alpha k^\alpha)_o}, \quad (6.1)$$

where u_α are the four-velocity vectors of the emitter and of the observer, and k^α is the affinely parametrised tangent vector field to the ray. In our case, both the emitter and the observer will be assumed to comove with the cosmic matter, so $u_\alpha = \delta^0_\alpha$, and then (6.1) simplifies to $1+z = k_e^t/k_o^t$. A further simplification results when the affine parameter is rescaled so that (5.8) holds; then

$$1+z = -k_e^t. \quad (6.2)$$

Since $k^t = -dt/d\lambda$ along a past-directed ray, using (6.2) we get from (5.7)

$$(1+z)^2 = \left\{ \frac{\mathcal{N}^2 (k^r)^2}{1+2E(r)} + \left(\frac{\Phi}{\mathcal{E}} \right)^2 [(k^x)^2 + (k^y)^2] \right\}_e. \quad (6.3)$$

Denote

$$(k^x)^2 + (k^y)^2 \stackrel{\text{def}}{=} J^2. \quad (6.4)$$

Then we obtain from (5.5) – (5.6)

$$\begin{aligned} \frac{1}{J} \frac{dJ}{ds} + 2\frac{\Phi_{,t}}{\Phi} k^t + 2\frac{\Phi_{,r}}{\Phi} k^r \\ - \frac{\mathcal{E}_{,r}}{\mathcal{E}} k^r - \frac{\mathcal{E}_{,x}}{\mathcal{E}} k^x - \frac{\mathcal{E}_{,y}}{\mathcal{E}} k^y = L(s), \end{aligned} \quad (6.5)$$

where

$$\begin{aligned} L(s) &= L(t(s), r(s), x(s), y(s)) \\ &\stackrel{\text{def}}{=} -\frac{\mathcal{N}}{\Phi(1+2E)} \frac{(k^r)^2 \mathcal{E}^2}{J^2} \mathcal{D}_k + \frac{\mathcal{E}_{,r}}{\mathcal{E}} k^r, \end{aligned} \quad (6.6)$$

$$\mathcal{D}_k \stackrel{\text{def}}{=} \left[\left(\frac{\mathcal{E}_{,r}}{\mathcal{E}} \right)_{,x} k^x + \left(\frac{\mathcal{E}_{,r}}{\mathcal{E}} \right)_{,y} k^y \right]. \quad (6.7)$$

As can be verified using (2.2), $(\mathcal{E}_{,r}/\mathcal{E})_{,x}$ and $(\mathcal{E}_{,r}/\mathcal{E})_{,y}$ are finite for all values of x , y and r . Consequently, \mathcal{D}_k is finite at all points where k^x and k^y are finite.

Since Φ depends only on t and r , while $\mathcal{E}_{,t} = 0$, we have $\Phi_{,t} k^t + \Phi_{,r} k^r = d\Phi/ds$ and $\mathcal{E}_{,r} k^r + \mathcal{E}_{,x} k^x + \mathcal{E}_{,y} k^y = d\mathcal{E}/ds$. Assuming that the initial condition for (6.5) is given at the observation point, where $s = s_o$, the solution of (6.5) may be written as

$$J = \frac{\mathcal{E}}{\mathcal{E}_o} \left(\frac{\Phi_o}{\Phi} \right)^2 J_o \exp \left(\int_{s_o}^s L(\lambda) d\lambda \right). \quad (6.8)$$

Substituting (6.8) in (6.3) we obtain

$$(1+z)^2 = \left[\frac{\mathcal{N}^2 (k^r)^2}{1+2E} + \frac{J_o^2 \Phi_o^4}{\mathcal{E}_o^2 \Phi^2} \exp \left(2 \int_{s_o}^s L(\lambda) d\lambda \right) \right]_e. \quad (6.9)$$

The geodesic would be radial if $J = J_o \equiv 0$, but such geodesics, as mentioned above, do not exist in general.

Now suppose we follow the ray from the observation point back in time to its intersection with the BB, where $\Phi \rightarrow 0$. Can (6.9) allow for infinite blueshift, i.e. for

$$\lim_{t \rightarrow t_B} z \stackrel{\text{def}}{=} z_{\text{BB}} = -1? \quad (6.10)$$

Both terms on the right-hand side of (6.9) are non-negative, so to allow $z_{\text{BB}} = -1$ they both must go to zero when $t \rightarrow t_B$. As follows from (4.7), the first term will go to zero when $dt_B/dr = 0$ at the intersection of the ray with the BB. But when $dt_B/dr \neq 0$ at that point, then a necessary condition for $z_{\text{BB}} = -1$ is

$$\lim_{t \rightarrow t_B} k^r = 0. \quad (6.11)$$

Nothing general can be said about the behaviour of the second term on the right-hand side of (6.9) when $\Phi_o = 0$, i. e. when the observation point is at the origin. In the L–T limit, a ray passing through the origin is radial, and the second term drops out. Here, however, L at the origin is infinite (because of $\Phi_{,r}/\Phi$, see (4.12)), so $\int_{s_o}^s L d\lambda$ can be infinite, too, and the exponential factor in (6.9) can compensate for $\Phi_o \rightarrow 0$. This can be investigated only numerically case by case.

If $\Phi_o \neq 0$, then, irrespectively of the behaviour of the first term, $1+z_{\text{BB}} \rightarrow \infty$ as long as the coefficient of Φ^{-2} in the last term has a nonzero limit at the BB. So, another necessary (but not sufficient) condition for $z_{\text{BB}} = -1$ is

$$\frac{J_o \Phi_o^2}{\mathcal{E}_o} \lim_{t \rightarrow t_B} \left[\frac{1}{\Phi} \exp \left(\int_{s_o}^s L(\lambda) d\lambda \right) \right] = 0. \quad (6.12)$$

From (3.4), $\mathcal{E} \neq 0$ everywhere, and is finite except at $x \rightarrow \infty$ and $y \rightarrow \infty$ (which are coordinate singularities). So, (6.12) can be fulfilled (1) when $J_o = 0$, or (2) when $\exp \left(\int_{s_o}^s L(\lambda) d\lambda \right) \xrightarrow[t \rightarrow t_B]{} 0$ faster than Φ .

Case (1) means that the null geodesic is orthogonal to the sphere of constant t and r at the *observation* point.

However, fulfilling (6.12) in this way is problematic: $L(\lambda)$ may be infinite at $\lambda = s_o$, and this may cause that $\int_{s_o}^s L(\lambda) d\lambda = \infty$ for any s_o . There are too many possibilities to identify a criterion for $z_{\text{BB}} = -1$ in this case. Moreover, this way of achieving $z_{\text{BB}} = -1$ would be unnatural: the second term in (6.9) would then vanish all along the ray between the observation point and the BB. The implication would be that $z_{\text{BB}} = -1$ if the ray is orthogonal to any single constant- (t, r) surface, independently of what happens between this surface and the BB.

A necessary condition for case (2) is

$$\lim_{t \rightarrow t_B} L(s) = +\infty \quad (6.13)$$

(because in integrating to the past $s < s_o$). Since $\mathcal{N} > 0$ (in consequence of the no-shell-crossing conditions [13]), and $1+2E \geq 0$ (to have the right signature), the sign of the infinity in L will be determined by the sign of \mathcal{D}_k , which can be any.

One of the ways of fulfilling (6.13) is $J \rightarrow 0$ at the BB (provided that k^r and \mathcal{D}_k do not go to zero too fast). But whether the ray from the BB is blue- or redshifted will depend here not only on the behaviour of J , k^r and dt_B/dr near the BB, but also on whether the mass-dipole component determined by $\mathcal{E}_{,r}/\mathcal{E}$ is increasing or decreasing as the (past-directed!) ray approaches the BB. When it decreases, $\mathcal{D}_k < 0$ and $\lim_{s \rightarrow s_{\text{BB}}} L(s) = +\infty$, in which case (because of $s < s_o$) $\exp \left(\int_{s_o}^s L(\lambda) d\lambda \right) \rightarrow 0$, and $z_{\text{BB}} = -1$ is possible. But when it increases, $\mathcal{D}_k > 0$ and $\exp \left(\int_{s_o}^s L(\lambda) d\lambda \right) \rightarrow \infty$, preventing $z_{\text{BB}} = -1$.

Thus, unlike in the L–T case, formulating a clearcut criterion for infinite blueshift in the Szekeres models is not possible, and the remarks above can only be used as suggestions for numerical experiments.

TABLE I: Is $z = -1$ possible at the BB?

$\left. \frac{dt_B}{dr} \right _{\text{BB}} = 0$	$k_{\text{BB}}^r = 0$	$J_{\text{BB}} = 0$	P	I
	$k_{\text{BB}}^r \neq 0$	$J_{\text{BB}} \neq 0$	$z_{\text{BB}} = \infty$	II
$\left. \frac{dt_B}{dr} \right _{\text{BB}} \neq 0$	$k_{\text{BB}}^r = 0$	$J_{\text{BB}} = 0$	P	III
	$k_{\text{BB}}^r \neq 0$	$J_{\text{BB}} \neq 0$	$z_{\text{BB}} = \infty$	IV
	$k_{\text{BB}}^r = 0$	$J_{\text{BB}} = 0$	P	V
	$k_{\text{BB}}^r \neq 0$	$J_{\text{BB}} \neq 0$	P	VI
	$k_{\text{BB}}^r = 0$	$J_{\text{BB}} = 0$	$z_{\text{BB}} = \infty$	VII
	$k_{\text{BB}}^r \neq 0$	$J_{\text{BB}} \neq 0$	$z_{\text{BB}} = \infty$	VIII

By separately considering the various possibilities we arrive at Table I. The last column contains reference numbers of the cases. “P” stands for “possibly $z_{\text{BB}} = -1$ ”. In cases VII and VIII the $z_{\text{BB}} = \infty$ is created by the first term on the right-hand side of (6.9), and the behaviour of the second term is irrelevant. In the other cases, whether $z_{\text{BB}} = -1$ is possible or not depends on the limit at the BB of $[(\Phi_{,r}/\Phi)(k^r)^2 \mathcal{D}_k/J^2]$ inside $L(s)$, and also on the sign of \mathcal{D}_k (depending on this sign, the limit, if infinite, may be $+\infty$ or $-\infty$). Case V becomes an infinitely blueshifted radial ray in the L–T limit.

VII. NULL GEODESICS IN SYMMETRIC SUBCASES OF THE SZEKERES SPACETIMES

In special cases, first integrals of the geodesic equations (5.3) – (5.6) exist. One of them is when Q is constant; then there exist null geodesics along which $y = Q$ and $k^y = 0$.² Equation (5.6) is then fulfilled identically.

A. Null geodesics in the axially symmetric subcase

Another special case is when P and Q are constant [15, 17]. Then the Szekeres spacetime is axially symmetric around $(x, y) = (P, Q)$, and a family of null geodesics exists on which $x = P$ and $y = Q$. The transformation

$$x = x' + P, \quad y = y' + Q \quad (7.1)$$

has then the same result as if

$$P = Q = 0, \quad (7.2)$$

which we shall assume. Then we introduce

$$x' = u \cos \varphi, \quad y' = u \sin \varphi, \quad (7.3)$$

which changes (2.1) and (2.2) to

$$ds^2 = dt^2 - \frac{\mathcal{N}^2 dr^2}{1 + 2E(r)} - \left(\frac{\Phi}{\mathcal{E}}\right)^2 (du^2 + u^2 d\varphi^2), \quad (7.4)$$

$$\mathcal{E} = \frac{1}{2S} (u^2 + S^2); \quad (7.5)$$

the dipole equator $\mathcal{E}_{,r} = 0$ is now at $u = S$. In these coordinates, the geodesic equations for (7.4) – (7.5) are

$$\frac{dk^t}{d\lambda} + \frac{\mathcal{N}\mathcal{N}_{,t}}{1 + 2E} (k^r)^2 + \frac{\Phi\Phi_{,t}}{\mathcal{E}^2} [(k^u)^2 + u^2 (k^\varphi)^2] = 0, \quad (7.6)$$

$$\begin{aligned} \frac{dk^r}{d\lambda} + 2\frac{\mathcal{N}_{,t}}{\mathcal{N}} k^t k^r \\ + \left(\frac{\mathcal{N}_{,r}}{\mathcal{N}} - \frac{E_{,r}}{1 + 2E}\right) (k^r)^2 + 2\frac{u\Phi S_{,r}}{S\mathcal{E}^2 \mathcal{N}} k^r k^u \\ - \frac{\Phi}{\mathcal{E}^2} \frac{1 + 2E}{\mathcal{N}} [(k^u)^2 + u^2 (k^\varphi)^2] = 0, \end{aligned} \quad (7.7)$$

$$\begin{aligned} \frac{dk^u}{d\lambda} + 2\frac{\Phi_{,t}}{\Phi} k^t k^u - \frac{uS_{,r}\mathcal{N}}{S\Phi(1 + 2E)} (k^r)^2 + 2\frac{\mathcal{N}}{\Phi} k^r k^u \\ - \frac{u}{S\mathcal{E}} (k^u)^2 + u \left(\frac{u^2}{S\mathcal{E}} - 1\right) (k^\varphi)^2 = 0, \end{aligned} \quad (7.8)$$

$$\frac{dk^\varphi}{d\lambda} + 2\frac{\Phi_{,t}}{\Phi} k^t k^\varphi + 2\frac{\mathcal{N}}{\Phi} k^r k^\varphi + 2\left(\frac{1}{u} - \frac{u}{S\mathcal{E}}\right) k^u k^\varphi = 0. \quad (7.9)$$

² Note that the case when $P = \text{constant}$ and $x = P$ along the geodesic is equivalent to $Q = \text{constant}$ and $y = Q$ under the transformation $(x, y) = (y', x')$ accompanied by the relabeling $(P, Q) = (\tilde{Q}, \tilde{P})$, which does not change the metric.

The remark made at the end of Sec. V applies also here: the coefficients of $(k^r)^2$ and of k^r in (7.8) and (7.9) require special treatment. However, these equations become regular when the geodesic stays within the $\{\varphi = \text{constant}, u = 0\}$ surface. Such a geodesic intersects every space of constant t on the symmetry axis.

Equation (7.9) has the first integral:

$$k^\varphi u^2 \Phi^2 / \mathcal{E}^2 = J_0, \quad (7.10)$$

where J_0 is constant along each geodesic (not necessarily null). When (7.10) is substituted in (5.7) transformed to the (u, φ) coordinates, the following results:

$$(k^t)^2 = \frac{\mathcal{N}^2 (k^r)^2}{1 + 2E} + \left(\frac{\Phi}{\mathcal{E}}\right)^2 (k^u)^2 + \left(\frac{J_0 \mathcal{E}}{u \Phi}\right)^2. \quad (7.11)$$

At the observation point (5.8) applies, at the emission point (6.2) can be used.

Equations (7.11) and (6.2) show that for geodesics emitted at the BB, where $\Phi = 0$, the observed redshift is infinite when $J_0 \neq 0$. A necessary (but not sufficient) condition for $1 + z_o = 0$ is $J_0 = 0$, i.e. the ray must proceed within the hypersurface of constant φ .

B. Null geodesics in the L–T limit

The L–T model follows from (2.1) – (2.2) as the limit $P_{,r} = Q_{,r} = S_{,r} = 0$ ($\implies \mathcal{E}_{,r} = 0$). Then the spheres of constant t and r become concentric, the spacetime becomes spherically symmetric and (5.5) – (5.6) imply

$$(k^x)^2 + (k^y)^2 = \frac{C^2 \mathcal{E}^2}{\Phi^4}, \quad (7.12)$$

where C is constant along the geodesic. Thus, the geodesic is radial ($k^x = k^y = 0$ along it) when $C = 0$.

When substituted in (6.3), Eq. (7.12) implies that along all nonradial rays $z \rightarrow \infty$ at the BB (where $\Phi \rightarrow 0$) irrespectively of whether $dt_B/dr = 0$ or not [5].

VIII. THE EXTREMUM REDSHIFT SURFACE

In the L–T limit, an Extremum Redshift *Hypersurface* (ERH) was defined. It is the locus where the redshift along past-directed *radial* rays achieves a local maximum or minimum. Since in L–T models the collection of all radial rays at any fixed time is two-parametric, the ERH is a 3-dimensional hypersurface in spacetime. But in a general Szekeres spacetime, radial directions are not defined. In the axisymmetric subcase, an analogue of the ERH exists, but, as will be seen below, it is 2-dimensional, so it will be called the Extremum Redshift *Surface*.

Consider a null geodesic that stays in the surface $\{u, \varphi\} = \{0, \text{constant}\}$; it obeys (7.8) and (7.9) identically. The remark made under (5.7) applies to it in an even stronger form: $k^r \neq 0$ at all points because with

$u = 0 = k^u = k^\varphi$ the geodesic would be timelike wherever $k^r = 0$. Assume the geodesic is past-directed and has its initial point at $r = 0$. Thus, r has to increase on it and can be used as a parameter. Using (6.2), we rewrite (7.6) as follows:

$$\frac{dz}{d\lambda} = \frac{\mathcal{N}\mathcal{N}_{,t}}{1+2E} k^r \frac{dr}{d\lambda}. \quad (8.1)$$

Changing the parameter to r we obtain

$$\frac{dz}{dr} = \frac{\mathcal{N}\mathcal{N}_{,t}}{1+2E} k^r. \quad (8.2)$$

Since $\mathcal{N} \neq 0$ from no-shell-crossing conditions [13] and $k^r > 0$, the extrema of z on such a geodesic occur where

$$\mathcal{N}_{,t} \equiv \Phi_{,tr} - \Phi_{,t} \mathcal{E}_{,r} / \mathcal{E} = 0. \quad (8.3)$$

Equation (8.3) was derived under the assumption $u = 0$. Thus, the set in spacetime defined by (8.3) is 2-dimensional (φ is constant, but arbitrary). We will call it Extremum Redshift Surface (ERS).

From (2.4) and (3.2) with $\Lambda = 0$ we obtain

$$\Phi_{,t} = r \sqrt{\frac{2M_0 r}{\Phi}} - k, \quad (8.4)$$

From now on we proceed assuming the function $E(r)$ in the same form as in Ref. [5],

$$2E(r) = -kr^2, \quad \text{where} \quad -k = 0.4 \quad (8.5)$$

(see Appendix C for the ERS equation without this simplification). Then $(3/2)E_{,r}/E - M_{,r}/M = 0$, and from (4.2) we obtain

$$\Phi_{,tr} = \sqrt{\frac{2M_0 r}{\Phi}} - k + \frac{M_0 r^3}{\Phi^2} t_{B,r}. \quad (8.6)$$

Using this, (8.4) and (7.5) with $u = 0$, Eq. (8.3) becomes

$$\sqrt{\frac{2M_0 r}{\Phi}} - k \left(1 - r \frac{S_{,r}}{S} \right) = -\frac{M_0 r^3}{\Phi^2} t_{B,r}. \quad (8.7)$$

To avoid shell crossings, $t_{B,r} < 0$ must hold [13], so the right-hand side of (8.7) is positive. The left-hand side is positive in consequence of (2.8) and (3.2).

With (8.5), $E > 0$ and we can use (A3) for Φ . Squaring both sides of (8.7) and remembering that $k < 0$ we obtain

$$(\cosh \eta + 1)(\cosh \eta - 1)^3 = -\frac{k^3 r^2 t_{B,r}^2}{M_0^2 (1 - r S_{,r}/S)^2}. \quad (8.8)$$

Denoting

$$\chi \stackrel{\text{def}}{=} \sinh^2(\eta/2) \quad (8.9)$$

Eq. (8.8) can be written as

$$\chi^4 + \chi^3 = -k^3 \left[\frac{r t_{B,r}}{4M_0 (1 - r S_{,r}/S)} \right]^2. \quad (8.10)$$

The ERH equation in Ref. [18] follows from (8.10) as the limit $S_{,r} = 0$. With $k < 0$, (8.10) is solvable for χ at any r , since its left-hand side is independent of r and can vary from 0 to $+\infty$ while the right-hand side is positive.

IX. AN EXEMPLARY AXIALLY SYMMETRIC QSS MODEL

Since, so far, it turned out to be impossible to determine the rays with infinite blueshift by exact calculations, we shall now attempt to detect them numerically in the Szekeres spacetimes given by (7.4) – (7.5). In choosing a simple form for $S(r)$ one must take care to obey (2.6) and (2.7), which, using (3.2) and (8.5), imply

$$1/r > S_{,r}/S. \quad (9.1)$$

Equations (9.1) and (3.4) will be fulfilled when

$$S = \sqrt{a^2 + r^2}, \quad (9.2)$$

where $a > 0$ is a constant. With (7.2), the equation of the dipole “equator” $\mathcal{E}_{,r} = 0$ becomes

$$x^2 + y^2 = S^2, \quad (9.3)$$

and the axis of symmetry is $x = y = 0$.

To define a model completely we need to prescribe the bang-time function $t_B(r)$. We choose it in the form

$$t_B(r) = \begin{cases} A \left(e^{-\alpha r^2} - e^{-\alpha r_b^2} \right) + t_{\text{BB}} & \text{for } r \leq r_b, \\ t_{\text{BB}} & \text{for } r \geq r_b, \end{cases} \quad (9.4)$$

where A, α, r_b and t_{BB} are constants. For $r \geq r_b$ this spacetime goes over into the Friedmann spacetime (see Appendix D), albeit represented in exotic coordinates.

Figure 1 shows the cross-section of the spacetime by a surface of (any) constant φ and $t = t_o = 1.2$ (most rays considered further on will have their initial points at this t). Each such surface consists of non-concentric circles, but is not flat, so Fig. 1 is not an isometric image.

A definition of the radius of each circle is not self-evident. From (2.1) it is seen that the radius can be defined either as the curvature radius $\Phi(t_o, r)$ or as a geodesic radius, by integrating $\sqrt{|g_{rr}|}$. But the value of the integral depends on the direction in the (x, y) surface, and the centers of circles of different radii do not coincide. The most natural definition seems to be

$$\mathcal{R}(r_f) \stackrel{\text{def}}{=} \int_0^{r_f} \frac{\Phi_{,r}(t_o, r)}{\sqrt{1+2E(r)}} dr \quad (9.5)$$

for two reasons:

(1) This path of integration goes along the dipole equator $\mathcal{E}_{,r} = 0$, so such \mathcal{R} is the same as in an L–T model.

(2) \mathcal{R} as defined by (9.5) coincides with

$$\tilde{\mathcal{R}} \stackrel{\text{def}}{=} \int_0^{r_f} \frac{\Phi_{,r}(t_o, r) - \Phi(t_o, r) \mathcal{E}_{,r} / \mathcal{E}}{\sqrt{1+2E(r)}} dr \quad (9.6)$$

averaged over all directions, i.e. over the whole sphere $r = r_f$. From (2.1), the surface element of such a sphere is $[\Phi(t_o, r_f)/\mathcal{E}(r_f, x, y)]^2 dx dy$, its surface area is $4\pi\Phi^2(t_o, r_f)$, and so the average of $\tilde{\mathcal{R}}$ over this sphere is

$$\langle \tilde{\mathcal{R}} \rangle = \mathcal{R} \quad (9.7)$$

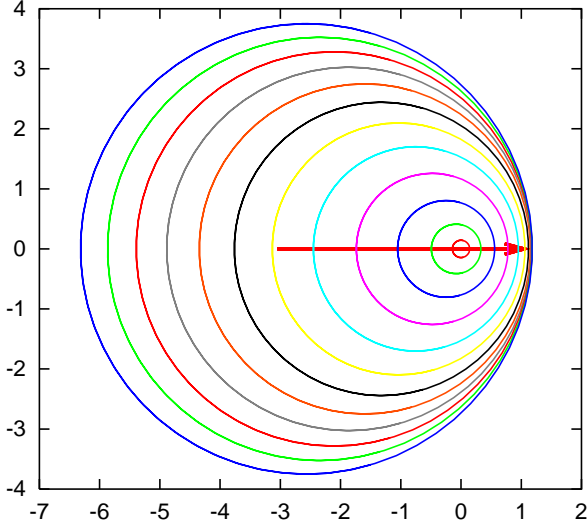


FIG. 1: The constant- r circles in the (r, u) surface mapped into a plane so that the smallest distances between them are the same as in the metric (7.4). The arrow points along the direction of the dipole maxima, $u = 0$. See text for more explanations.

because

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{\mathcal{E}_{,r}}{\mathcal{E}^3} = 0, \quad (9.8)$$

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{\mathcal{E}^2} = 4\pi. \quad (9.9)$$

Equations (9.7) – (9.9) apply with any \mathcal{E} given by (2.2).

The r -coordinates of the circles in Fig. 1 run from $r_0 = 0$ to $r_{12} = 2.4$ at intervals of $\Delta r = 0.2$. Their radii were calculated from (9.5) with M , E and \mathcal{E} given by (3.2), (8.5) and (7.5), respectively. The distances between them were calculated along the dipole maximum ($u = 0$), from

$$d_{\max} = \int_{r_i}^{r_{i+1}} \frac{\Phi_{,r}(t_o, r) - \Phi(t_o, r)S_{,r}/S}{\sqrt{1 + 2E(r)}} dr, \quad (9.10)$$

$i = 0, 1, \dots, 12,$

because $(\mathcal{E}_{,r}/\mathcal{E})|_{u=0} = S_{,r}/S$. The three largest circles are in the Friedmann region.³

Figure 1 is drawn so that the shortest distances between the circles are the same as the d_{\max} in (9.10). If the circles were drawn so that the longest distances between them were the same as along the dipole minimum in the curved surface (i.e. with $+$ in the numerator of (9.10)), the image would be the same.

³ In the Friedmann region, the circles are non-concentric in consequence of the coordinate choice, see Appendix D.

X. AXIAL RAYS IN THE AXIALLY SYMMETRIC QSS MODEL

For the numerical examples we chose

$$(A, \alpha, t_{\text{BB}}, r_b) = (1, 2, 0, 2). \quad (10.1)$$

The first calculation was for two null geodesics going back in time with $u = 0 = k^\varphi$ from $(r, t) = (0, t_B(0) + 0.1)$ and $(r, t) = (0, t_B(0) + 0.2)$, respectively, in the model with $a^2 = 0.1$, see Fig. 2. The numerical calculation confirmed that $z \rightarrow -1$ as the rays approach the BB, see Table II. This model belongs to case V in Table I.

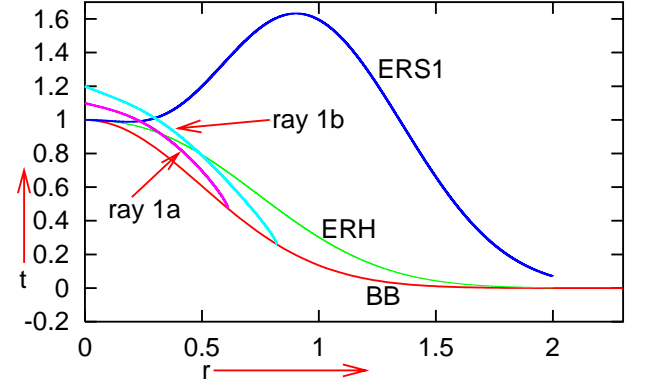


FIG. 2: The $t(r)$ graphs of two rays approaching the Big Bang given by (9.4) within the surface $\{u, \varphi\} = \{0, \text{constant}\}$ in the metric (7.4) – (7.5). See text for more explanation.

Figure 2 shows the BB profile, the $t(r)$ graphs of the two rays and the ERS profile corresponding to $a^2 = 0.1$. The ERH profile in the L–T spacetime with the same $t_B(r)$ and $E(r)$ is also included. It can be seen that in a Szekeres model, the rays enter the ERS (i.e. begin acquiring negative contributions to redshift) at smaller r than in the corresponding L–T model. Note that, in consequence of the mass dipole, the geometrical distance between r_1 and r_2 along a line of constant (t, u, φ) and $\mathcal{E}_{,r} > 0$ is shorter in a Szekeres spacetime than in the corresponding L–T spacetime. Thus, the figure is not a faithful image of the geometrical relations.

The jump in the ERS profile at $r = 2$ is a consequence of the jump in dt_B/dr at $r = r_b = 2$; see (9.4) and (8.10). Similar jumps will be seen in other ERS profiles in the next figures. The ERH in Fig. 2 also has a jump at $r = 2$, but, with the parameter values given by (10.1) and $a^2 = 0.1$, the right-hand side of (8.10) is 1681 times smaller in an L–T model than in the Szekeres model, so the jump in χ is also much smaller.

Figure 3 demonstrates the influence of the value of a on the shape of the ERS. Ray 1b and ERS1 are the same as in Fig. 2. Ray 2 and ERS2 were calculated with $a^2 = 0.07$, ERS3 was calculated with $a^2 = 0.04$; the values of the other parameters were the same as for ray 1b. As seen, smaller a gives a stronger effect. Larger a produces

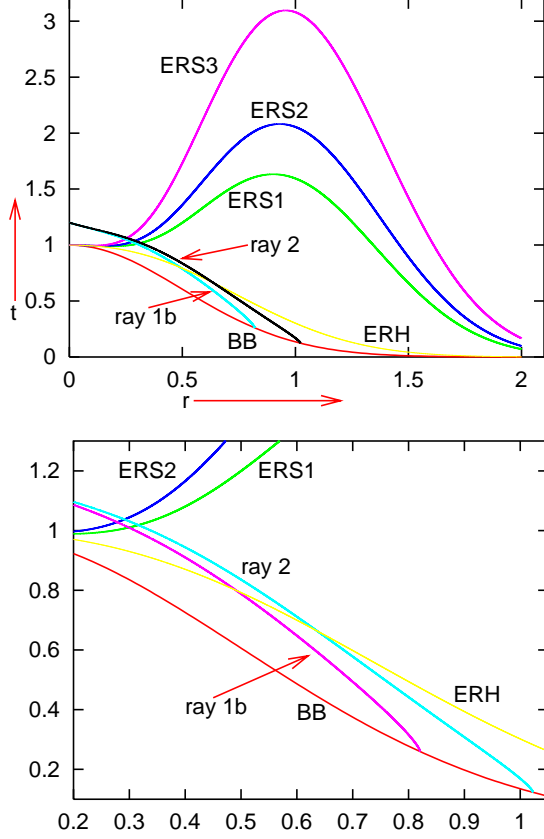


FIG. 3: **Upper panel:** Smaller value of a results in a higher maximum of the ERS. See text for more explanation. **Lower panel:** Closeup view on the rays; the ERS3 is omitted.

a smaller difference between the ERS and the ERH of the corresponding L-T model. In the limit $a \rightarrow \infty$ the L-T result would be recovered – as can be seen from (9.2) and (8.10), this limit has the same effect as $S_{,r} = 0$. The added flexibility in the Szekeres models is that the time of flight of the ray under the ERS can be increased by manipulating the S function. In the L-T models, increasing this time was possible only by manipulating the BB profile. The lower panel of Fig. 3 is a closeup view on the region where many lines intersect.

Figure 4 shows the comparison of rays 1a and 1b from Fig. 2 with the rays in the L-T model that have the same initial (r, t) ; the BB profile is the same in both models. The rays in the Szekeres model hit the BB at larger values of r . In the L-T model, increasing the r coordinate of the intersection of the ray with the BB required moving up the initial point of the ray, in the Szekeres model this can be achieved by increasing $S_{,r}/S$ without changing the initial data of the ray.

Figure 5 shows the redshift profiles along the rays seen in Fig. 4. The maximum redshift along each Szekeres ray is smaller than the maximum along the corresponding L-T ray. This is consistent with the message of Fig. 2: the

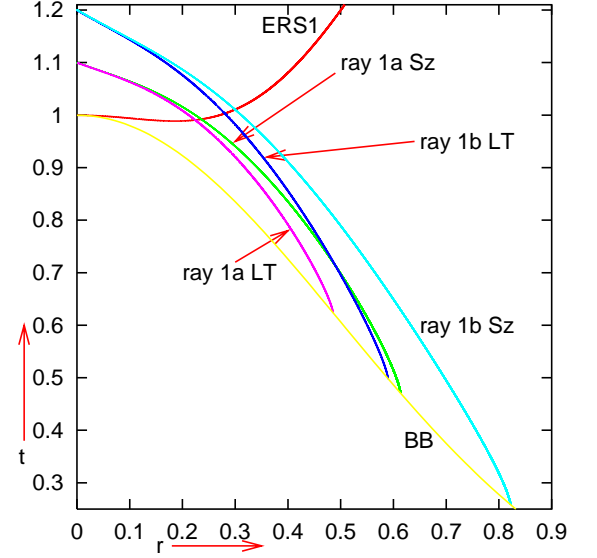


FIG. 4: Rays 1a and 1b from Fig. 2 compared with their L-T counterparts. See text for more explanation.

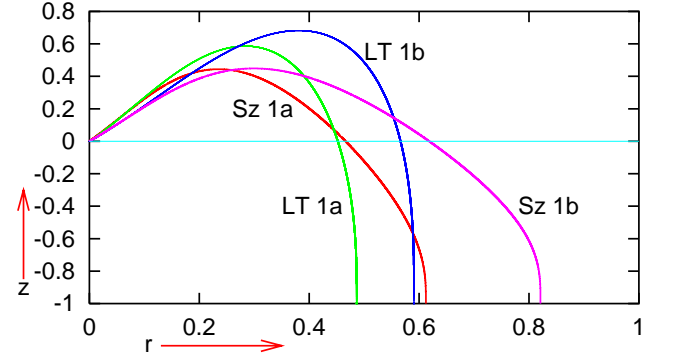


FIG. 5: Redshift profiles along the rays from Fig. 2 compared with the corresponding redshift profiles in the L-T model. See text for more explanation. At the intersections of the rays with the Big Bang all redshifts approach -1 .

Szekeres rays begin acquiring negative contributions to redshift at smaller r , so the maximum z is smaller for the Szekeres rays than for the corresponding L-T rays.

XI. OTHER RAYS WITH BLUESHIFT IN THE AXIALLY SYMMETRIC CASE

As seen from (7.11), a necessary condition for the existence of blueshift along a ray emitted at the BB is $J_0 = 0$, i.e. the ray must proceed within a hypersurface of constant φ . In this section, we will maintain this assumption, but will relax the assumption $u = 0$ along the ray that was adopted in Sec. X. Then (7.9) is still fulfilled identically. As remarked at the end of Sec. V, the coefficients \mathcal{N}/Φ in front of $(k^r)^2$ and k^r in (7.8) become infinite at $r = 0$. Therefore, the limits at $r \rightarrow 0$ of these whole

TABLE II: Parameters of the rays from Fig. 2

Parameter	Ray 1a (lower)	Ray 1b (upper)
t at $r = 0$	$t_B(0) + 0.1$	$t_B(0) + 0.2$
r at the BB	0.61238227292746328	0.82090257143313361
t at the BB	0.47201970792080822	0.25948561090989042
$1 + z$ at the BB	$2.53722758372994317 \times 10^{-6}$	$2.63658970817360853 \times 10^{-6}$
maximum z	0.44409843877390864	0.44862603680491642
r at maximum z	0.23503329565169392	0.29909096252077694
t at maximum z	0.99199314244492787	1.0102113079060080
r at $z = 0$	$r = 0.46613259040713573$	0.61839965892230220
t at $z = 0$	0.74684282247031808	0.62045744911301359

terms have to be calculated exactly. In the calculation below we assume that at $r = 0$ both k^r and k^u are finite and that the calculation is done at $t > t_B$.

With S chosen as in (9.2) we have $\lim_{r \rightarrow 0} S_{,r} = 0$ and $S(0) = a > 0$. Knowing this we obtain using (4.11)

$$\lim_{r \rightarrow 0} \frac{S_{,r}}{\Phi} = \frac{1}{aR} < \infty, \quad (11.1)$$

so the third term in (7.8) is finite at $r = 0$.

For calculating the limit at $r \rightarrow 0$ of the fourth term in (7.8) we observe that it can be finite only when either $\lim_{r \rightarrow 0} k^r = 0$ or $\lim_{r \rightarrow 0} k^u = 0$. However, with $J_0 = 0 = k^r(0)$, Eq. (7.11) becomes a contradiction with (5.8) at $r = 0$, so $\lim_{r \rightarrow 0} k^r \neq 0$. Consequently,

$$\lim_{r \rightarrow 0} k^u = 0. \quad (11.2)$$

We now calculate $\lim_{r \rightarrow 0} (k^u/\Phi)$ along a null geodesic. We begin with the de l'Hôpital rule and use (7.8) for $dk^u/d\lambda$. Then we use $k^\varphi = 0$ and (11.1). The result is

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{k^u}{\Phi} &= \lim_{r \rightarrow 0} \frac{dk^u/d\lambda}{\Phi_{,t} k^t + \Phi_{,r} k^r} = \\ &= - \lim_{r \rightarrow 0} \left\{ \frac{1}{\Phi_{,r} k^r} \left[\frac{\Phi_{,t}}{\Phi} k^t k^u - \frac{u\mathcal{N}}{aRS(1+2E)} (k^r)^2 \right. \right. \\ &\quad \left. \left. + 2\mathcal{N}k^r \frac{k^u}{\Phi} - \frac{u(k^u)^2}{S\mathcal{E}} \right] \right\}. \end{aligned} \quad (11.3)$$

In this, we use (4.10), (11.2), (7.11) with $J_0 = 0$ taken at the observation point where $k^t = -1$ and $\Phi = 0$, (5.2), (4.11), (8.5) and (9.2). Solving the result for $\lim_{r \rightarrow 0} (k^u/\Phi)$ we obtain

$$\lim_{r \rightarrow 0} \frac{k^u}{\Phi} = \frac{u}{3a^2R^2}, \quad (11.4)$$

so the fourth term in (7.8) is also finite at the origin.

Since numerical calculations cannot respect such intricate limits automatically, the initial points for geodesics passing through the origin must be chosen at the origin, as remarked at the end of Sec. V.

For the next experiments, we take (past-directed) null geodesics with $k^\varphi \equiv 0$ and initial $u_o > 0$. Note, from (7.8), that if $u_o \neq 0$ then $(dk^u/d\lambda)_o \neq 0$ even if $k_o^u = 0$, so the ray will not stay in a constant- u surface. We consider the following cases:

- (I) $u_o = S/100$. This is close to the case $u = 0$ investigated in Sec. X, where $\mathcal{E}_{,r}/\mathcal{E}$ had a maximum equal to $S_{,r}/S$.
- (II) $u_o = S/10$.
- (III) $u_o = S/2$.
- (IV) $u_o = S$, where the contribution of the dipole is zero, $\mathcal{E}_{,r} = 0$.
- (V) $u_o = 2S$.
- (VI) $u_o = 10S$.
- (VII) $u_o = 100S$.
- (VIII) $u_o \rightarrow \infty$, where the dipole contribution has the minimum equal to $(-S_{,r}/S)$.

For $u > S$ the metric and the Christoffel symbols will have to be transformed to the new coordinate $w = 1/u$. This is because with u becoming very large, k^u tends to ∞ much faster than u and stops the program before the calculation comes near to the BB.

In every case, the (t, r) coordinates of the initial point will be the same as for ray 1b in the previous examples,

$$(t, r)_o = (t_B(0) + 0.2, 0). \quad (11.5)$$

At the initial point, where $\Phi = 0$, we use (7.11) with $J_0 = 0$, (5.8), (3.6) – (3.7), (3.2), (8.5) and (A3) to find

$$k_o^r = \frac{-k}{M_0(\cosh \eta_o - 1)}, \quad (11.6)$$

where η_o is at $t = t_B(0) + 0.2$. Next, using (11.4), (11.1), (4.11), (8.5) and (11.6) we obtain from (7.8)

$$\left. \frac{dk^u}{d\lambda} \right|_o = \frac{uk^2}{3aSM_0^2(\cosh \eta - 1)^2} \Big|_o. \quad (11.7)$$

In the transformed coordinate $w = 1/u$, the metric and the function \mathcal{N} are replaced by

$$ds^2 = dt^2 - \frac{\tilde{\mathcal{N}}^2 dr^2}{1 + 2E(r)} - \left(\frac{\Phi}{\tilde{\mathcal{E}}}\right)^2 (dw^2 + w^2 d\varphi^2), \quad (11.8)$$

$$\tilde{\mathcal{N}} \stackrel{\text{def}}{=} \Phi_{,r} - \Phi \tilde{\mathcal{E}}_{,r} / \tilde{\mathcal{E}}, \quad (11.9)$$

where

$$\tilde{\mathcal{E}} = \frac{w^2 S}{2} + \frac{1}{2S}, \quad (11.10)$$

and the equator of the dipole is at $w = 1/S$. The only changes in (7.6), (7.10) and (7.11) are $(u, \mathcal{N}, \mathcal{E}) \rightarrow (w, \tilde{\mathcal{N}}, \tilde{\mathcal{E}})$, while (7.7) – (7.9) change to

$$\begin{aligned} \frac{dk^r}{d\lambda} + 2 \frac{\tilde{\mathcal{N}}_{,t}}{\tilde{\mathcal{N}}} k^t k^r \\ + \left(\frac{\tilde{\mathcal{N}}_{,r}}{\tilde{\mathcal{N}}} - \frac{E_{,r}}{1 + 2E} \right) (k^r)^2 - 2 \frac{w \Phi S_{,r}}{S \tilde{\mathcal{E}}^2 \tilde{\mathcal{N}}} k^r k^w \\ - \frac{\Phi}{\tilde{\mathcal{E}}^2} \frac{1 + 2E}{\tilde{\mathcal{N}}} [(k^w)^2 + w^2 (k^\varphi)^2] = 0, \end{aligned} \quad (11.11)$$

$$\begin{aligned} \frac{dk^w}{d\lambda} + 2 \frac{\Phi_{,t}}{\Phi} k^t k^w + \frac{w S_{,r} \tilde{\mathcal{N}}}{S \Phi (1 + 2E)} (k^r)^2 + 2 \frac{\tilde{\mathcal{N}}}{\Phi} k^r k^w \\ - \frac{w S}{\tilde{\mathcal{E}}} (k^w)^2 + w \left(\frac{w^2 S}{\tilde{\mathcal{E}}} - 1 \right) (k^\varphi)^2 = 0, \end{aligned} \quad (11.12)$$

$$\frac{dk^\varphi}{d\lambda} + 2 \frac{\Phi_{,t}}{\Phi} k^t k^\varphi + 2 \frac{\tilde{\mathcal{N}}}{\Phi} k^r k^\varphi + 2 \left(\frac{1}{w} - \frac{w S}{\tilde{\mathcal{E}}} \right) k^w k^\varphi = 0. \quad (11.13)$$

Also, (11.4) and (11.7) change to:

$$\lim_{r \rightarrow 0} \frac{k^w}{\Phi} = -\frac{w}{3a^2 R^2}, \quad (11.14)$$

$$\left. \frac{dk^w}{d\lambda} \right|_o = -\frac{wk^2}{3aSM_0^2(\cosh \eta - 1)^2} \Big|_o. \quad (11.15)$$

Figure 6 shows the $t(r)$ graphs of rays I – VIII compared with the $t(r)$ graph of ray 1b. Ray I very nearly coincides with ray 1b. As u_o increases, the ray hits the BB at ever smaller r . With $u_o > S$ the graphs become ever closer to each other. The lower end of ray V is barely visible at the BB, rays VI – VIII coincide at the scale of the figure. Ray VIII, similarly to ray 1b, proceeds within the surface of constant φ and u , but with $u = \infty$ ($w = 0$).

Only rays 1b and VIII run within the (t, r) surface shown in Fig. 6. Along the other rays u varies, so this figure shows the projections of those rays on the (t, r) surface along lines of constant (t, r) .

Figure 7 shows the projections of rays I – VII on the (u, r) surface along the lines of constant u and r . Curves I to IV, V_u and VI_u are the $u(r)$ functions for the corresponding rays, curves V_w and VI_w are the $w(r) = 1/u(r)$ functions. Such a change of variables was necessitated by

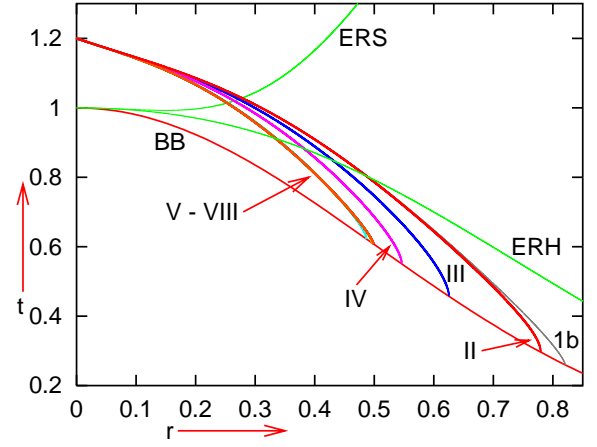


FIG. 6: The $t(r)$ graphs of rays II – VIII compared with the graph for ray 1b. Ray I nearly coincides with ray 1b and is omitted. Ray IV has its initial point at the equator of the dipole. See text for more explanation.

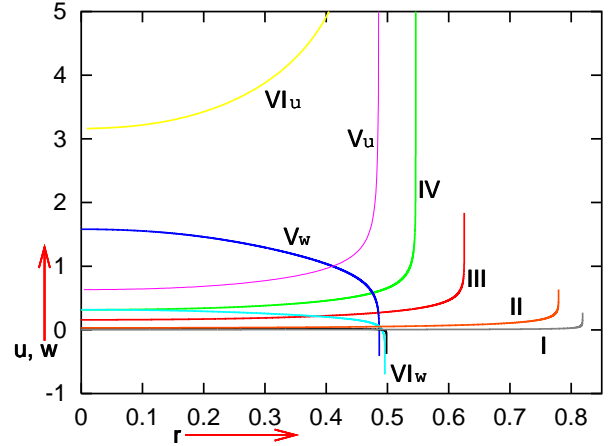


FIG. 7: The $u(r)$ and $w(r)$ graphs for rays I – VII. See text for more explanation.

numerical problems, as mentioned above. The $w(r)$ curve for ray VII is barely visible between I and II from $r = 0$ to nearly $r = 0.5$, then it turns down and nearly coincides with curve VI_w . The corresponding $u(r)$ curve lies far above the upper margin of the figure. Rays 1b and VIII proceed along the line $u = 0$. The rays that nearly coincide in Fig. 6 are widely separated in the u direction.

The rays that have small u_o stay close to the $u = 0$ line except near the end point. This has consequences for the redshift profile; see below.

Figure 8 shows the $t(u)$ and $t(w)$ graphs for rays I – VII. The labels follow the same rules as in Fig. 7. All rays have their upper ends at the same $t = 1.2$ because this is the t -coordinate of their initial points. As before, the $t(w)$ curve for ray VII is barely visible between I and II from $t = 1.2$ down to $t \approx 0.6$, then it turns left and nearly coincides with curve VI_w . The corresponding $t(u)$ curve lies far beyond the right margin of the figure. Rays 1b and VIII proceed along the line $u = 0$.

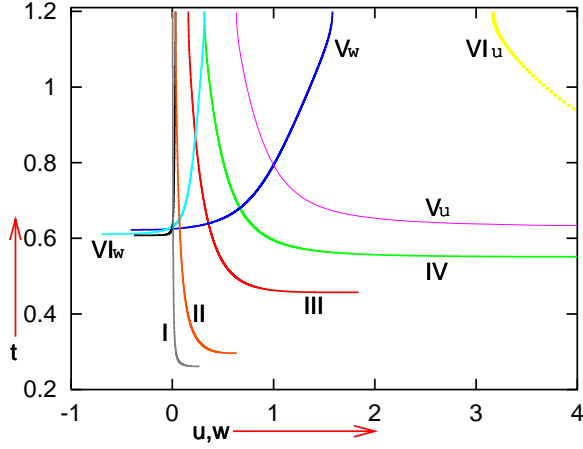


FIG. 8: The $t(u)$ and $t(w)$ graphs for rays I – VII. See text for more explanation.

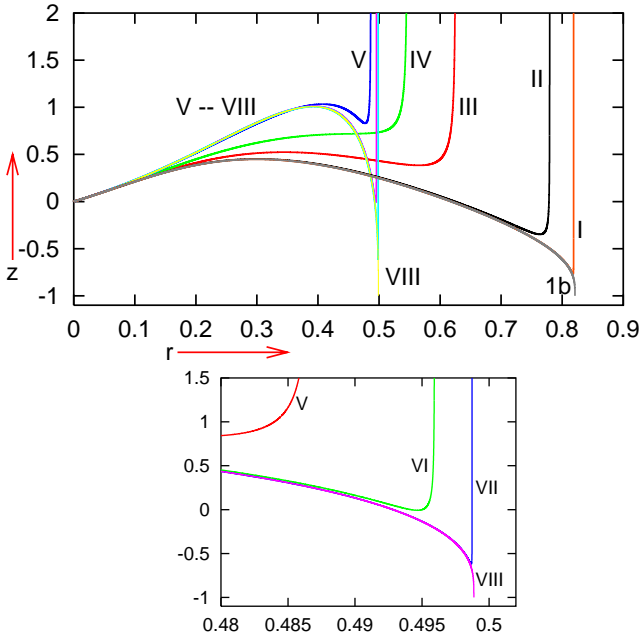


FIG. 9: **Upper panel:** The $z(r)$ redshift profiles for rays I – VIII. See text for more explanation. **Lower panel:** Enlarged view of profiles V to VIII in the r -range where they are close together in the upper panel.

Figure 9 shows the $z(r)$ profiles for all the rays. Only rays 1b and VIII have $z \approx -1$ at the BB (see Table III for the actual numerical values). Along the other rays z first achieves a maximum, then decreases, goes through a minimum and becomes very large near the BB. Table III shows the minima of $1 + z$ along the rays.

On rays I and VII, on which u_o is near its “axial” values $u = 0$ and $u = \infty$, respectively, the minimum z is near to -1 , and the graphs suggest that it is becoming still nearer to $z = -1$ when u approaches zero or infinity. This implies that on rays that go through the origin $r = 0$ with a sufficiently small (or large) value of

TABLE III: Local minima of $1 + z$ on the rays from Fig. 9

Ray	Minimum $1 + z$
1b	$2.53722758372994317 \times 10^{-6}$
I	0.23305708892873711
II	0.65207915516116299
III	1.3841852900906961
IV	1
V	1.8308297303885050
VI	0.99266554207399438
VII	0.38155727836044029
VIII	$8.27466387436995242 \times 10^{-6}$

u , the point of minimum z may lie closer to the BB than the last scattering hypersurface (the Szekeres model does not apply before last scattering because of $p = 0$). Such a ray would thus display a finite blueshift to the present observer even if it were emitted during last scattering.

The meaning of the initial u on a ray going off from the origin becomes clearer when u is transformed, in a surface of constant r , by

$$u = S \tan(\vartheta/2). \quad (11.16)$$

Then the metric of a surface of constant t and r in (2.1) becomes $ds_2^2 = \Phi^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$. The value $u = 0$ corresponds to $\vartheta = 0$, the limit $u \rightarrow \infty$ corresponds to $\vartheta \rightarrow \pi$. Thus, the only null geodesics that can display infinite blueshifts to an observer at $r = 0$ are those that reach her tangentially to the directions $\vartheta = 0$ and $\vartheta = \pi$. Geodesics approaching this observer from any other direction will display finite blueshifts or, if emitted at the BB, infinite redshifts.

Figure 10 shows the projections of rays I – VIII on the surface of constant t and φ along the $r = \text{constant}$ lines. The coordinates of the graph are $X_1 = r \cos \vartheta$ and $X_2 = r \sin \vartheta$. The arrow within the figure is parallel to the half-line $u = \vartheta = 0$, on which the dipole component of mass density has maxima. The dotted line $X_2 = 0$ is the projection of the axial rays 1b (going from $(X_1, X_2) = (0, 0)$ to the right) and VIII (going from $(0, 0)$ to the left). The thin circles are curves of constant r .

For clarity, Fig. 10 does not show the mirror-reflections of the rays in the $X_2 = 0$ axis. One obtains the collection of rays running in all the $\varphi = \text{constant}$ hypersurfaces by rotating the plane of Fig. 10 around the $X_2 = 0$ axis. Thus, rays going off from $r = 0$ with initial $u > 0$ (i.e. $\vartheta > 0$) and $u < \infty$ ($\vartheta < \pi$) form a funnel around the direction $u = 0$; only the rays with $u = 0$ and $u = \infty$ remain axial all the way and display infinite blueshifts to the observer when they reach the BB.

Note that rays running close to the half-line $\vartheta = 0$ (i.e. $u = 0$) bend away from it when approaching the BB, while those running close to the half-line $\vartheta = \pi$ ($u = \infty$) bend toward it at the BB. We will see this pattern repeated in the nonsymmetric model in Sec. XII.

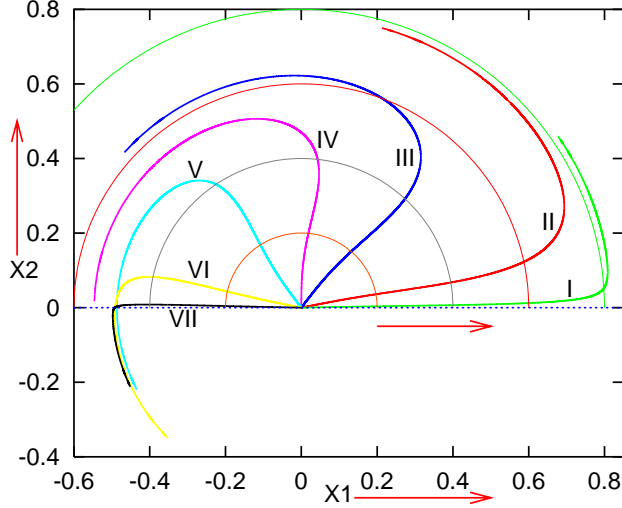


FIG. 10: Rays I – VIII projected along the $r = \text{constant}$ lines on the surface of constant t and φ in the polar coordinates (11.16). See explanations in the text.

Rays II – VII behave similarly to nonradial rays in an L–T model: when they approach the BB where $dt_B/dr \neq 0$, they bend sideways and hit the BB tangentially to a hypersurface of constant r (see Fig. 10 in Ref. [5]). But in L–T models any ray passing through the center $r = 0$ is radial, so all the rays shown in this figure would become radial in the L–T limit. Thus, the limiting transition from a Szekeres to an L–T model is discontinuous, just like the transition from L–T to Friedmann, in which all blueshifts discontinuously disappear.

Figures 9 and 10 show that the strongly blueshifted (SB) rays are centers of instability. Consider a single SB ray. The rays that are near to it develop negative z , but before they hit the BB, z goes through a minimum > -1 , and then increases to infinity. As the neighbouring rays move nearer to the SB ray, the minimum becomes closer to -1 , but z at the BB is still infinite, and the ray still ends up being tangent to the BB and to an $r = \text{constant}$ contour. Rays on opposite sides of the SB ray bend in opposite directions. Only the SB ray has $z = -1$ at the BB, and hits the BB orthogonally to an $r = \text{constant}$ contour. In the axisymmetric case we knew which rays would be SB, so we forced the numerical program to keep the ray on the SB path exactly. However, in the general case we do not know where the SB paths are, and this instability will render finding the SB paths numerically impossible: we can only approach them and observe the characteristic features described above; see Sec. XII.

In Fig. 10 the SB rays are orthogonal to the contours of constant r on approach to the BB, and the projections of the SB rays on the constant- (t, φ) hypersurface are tangent to the dipole axes at the contact with the BB. (They are in fact tangent to these axes all the way.) We will see in Sec. XII that the first property is nearly reproduced in the nonsymmetric QSS model, while the

second one does not survive.

XII. NULL GEODESICS IN A SIMPLE EXAMPLE OF A GENERAL QSS MODEL

We shall now consider null geodesics going off from the origin to the past in a simple QSS model that has no symmetry. In order to keep calculations simple, and to stay close to the axially symmetric example of Sec. IX, we still assume M , E , S and t_B of the forms (3.2), (8.5), (9.2) and (9.4), respectively. We choose $P(r)$ and $Q(r)$ so that (2.6) and (2.7) are fulfilled in the simplest way possible. These two inequalities, with (3.2), (8.5) and (9.2) are equivalent to the following one:

$$(P_{,r})^2 + (Q_{,r})^2 < \frac{a^2}{r^2} + \frac{a^2}{a^2 + r^2}. \quad (12.1)$$

In order to avoid any symmetries, (P, Q, S) must be linearly independent. We choose

$$P(r) = \frac{pa}{2(a^2 + r^2)}, \quad Q(r) = \frac{qa}{\sqrt{a^2 + r^2}}, \quad (12.2)$$

where p and q are constant parameters. The functions (P, Q, S) were chosen such that $P_{,r}(0) = Q_{,r}(0) = S_{,r}(0) = 0$. This is needed to cancel the infinity in \mathcal{N}/Φ at $r = 0$ in (5.5) and (5.6) by $(\mathcal{E}_{,r}/\mathcal{E})_{,x}$ and $(\mathcal{E}_{,r}/\mathcal{E})_{,y}$. Equation (12.1) will be obeyed at all r if

$$p^2 < 4a^4, \quad q^2 < 5a^2; \quad (12.3)$$

see Appendix E for a proof. Since in most of the examples so far we had $a^2 = 0.1$, we now choose

$$(a^2, p, q) = (0.1, 0.15, 0.6). \quad (12.4)$$

At any chosen r , the extrema of the dipole component of the mass-density (see comment under (2.8)) occur where $(\mathcal{E}_{,r}/\mathcal{E})_{,x} = 0$ and $(\mathcal{E}_{,r}/\mathcal{E})_{,y} = 0$, i.e. where

$$\begin{aligned} x - P &= -SP_{,r}/\mathcal{W}, & y - Q &= -SQ_{,r}/\mathcal{W}, \\ \mathcal{W} &\stackrel{\text{def}}{=} S_{,r} + \varepsilon \sqrt{(P_{,r})^2 + (Q_{,r})^2 + (S_{,r})^2}, \\ \varepsilon &\stackrel{\text{def}}{=} \pm 1. \end{aligned} \quad (12.5)$$

The value of $\mathcal{E}_{,r}/\mathcal{E}$ at these extrema is

$$(\mathcal{E}_{,r}/\mathcal{E})_{\text{ex}} = \varepsilon \sqrt{(P_{,r})^2 + (Q_{,r})^2 + (S_{,r})^2}/S, \quad (12.6)$$

and so $\varepsilon = +1$ corresponds to the maximum of $\mathcal{E}_{,r}/\mathcal{E}$, while $\varepsilon = -1$ corresponds to the minimum.

The positions of the $r = \text{constant}$ spheres in a $t = \text{constant}$ space are illustrated in Figs. 11 – 13. The values of the r -coordinate on the spheres range from $r = 0.2$ to $r = 3$ at intervals of $\Delta r = 0.2$. The five largest spheres are in the Friedmann region. The family of spheres was first mapped into an Euclidean space in such a way that the shortest Euclidean distances between

them are the same as the shortest geodesic distances in the Szekeres spacetime; see Sec. IX. These figures represent projections of the spheres on the (X_1, X_2) , (X_1, X_3) and (X_2, X_3) Euclidean coordinate planes. The centers of the spheres are marked with dots, the positions of the dipole maxima on the spheres are marked with crosses. The crosses do lie on the spheres, but not on their outer edges seen from the three directions. This is why they project into the interiors of the great circles, which is most conspicuous in Fig. 11. See Appendix F for information on the calculations underlying Figs. 11 – 13.

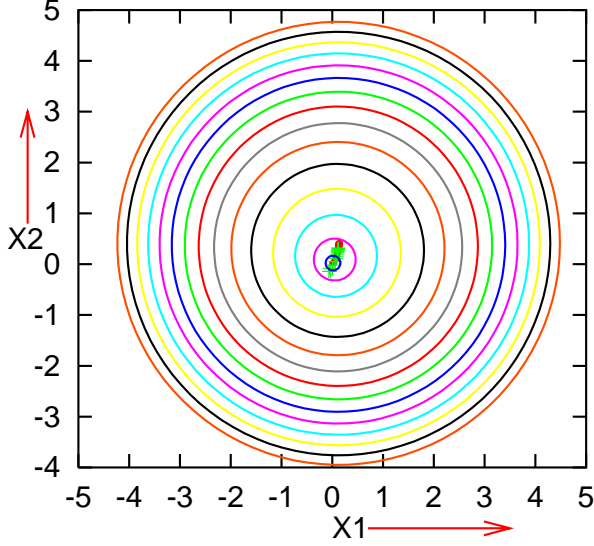


FIG. 11: Spheres of constant (t, r) in the nonsymmetric Szekeres model (2.1) – (2.3) with P , Q and S given by (12.2), (12.4) and (9.2). This is a projection on the (X_1, X_2) Cartesian plane. See explanation in text.

Using (4.11) we find

$$\lim_{r \rightarrow 0} \left(\frac{P_{,r}}{\Phi} \right) = -\frac{p}{a^3 R}, \quad \lim_{r \rightarrow 0} \left(\frac{Q_{,r}}{\Phi} \right) = -\frac{q}{a^2 R}. \quad (12.7)$$

From (12.7) and (11.1) it follows that the coefficients of $(k^r)^2$ in (5.5) and (5.6) have finite limits at $r \rightarrow 0$, so the ones to take care about are the terms containing $k^r k^x$ and $k^r k^y$. Equation (5.7) implies that $k^r \neq 0$ at $r = 0$ because otherwise also $k^t = 0$ at $r = 0$, and the geodesic would be spacelike at this point. This means that the terms in question can be finite at the origin only if $k^x = k^y = 0$ there. Thus we can apply the de l'Hôpital rule to calculate the limits at $r \rightarrow 0$ of k^x/Φ and k^y/Φ . We do it in the same way as in (11.3): we calculate these limits along the ray, and in the second step use (5.5) and (5.6) to substitute for $d(k^x)/d\lambda$ and $d(k^y)/d\lambda$. The results are, using (4.9) – (4.11):

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{k^x}{\Phi} &= \frac{1}{3R} \lim_{r \rightarrow 0} \frac{\mathcal{E}_{,r} \mathcal{E}_{,x} - \mathcal{E} \mathcal{E}_{,rx}}{\Phi} \\ &= \frac{1}{3a^2 R^2} \left\{ x - \frac{3p}{4a} + \frac{q}{a^2} \left(x - \frac{p}{2a} \right) (y - q) \right. \end{aligned}$$

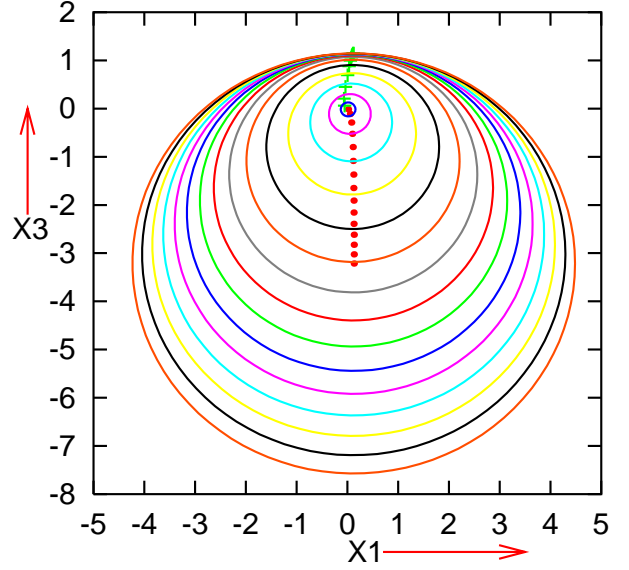


FIG. 12: The same spheres as in Fig. 11 projected on the (X_1, X_3) Cartesian plane.

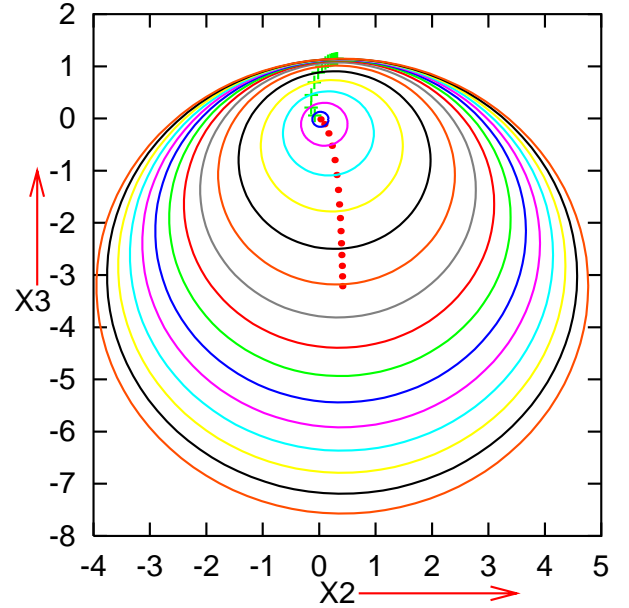


FIG. 13: The same spheres as in Fig. 11 projected on the (X_2, X_3) Cartesian plane.

$$+ \frac{p}{4a^3} \left[\left(x - \frac{p}{2a} \right)^2 - (y - q)^2 \right] \Big\}, \quad (12.8)$$

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{k^y}{\Phi} &= \frac{1}{3R} \lim_{r \rightarrow 0} \frac{\mathcal{E}_{,r} \mathcal{E}_{,y} - \mathcal{E} \mathcal{E}_{,ry}}{\Phi} \\ &= \frac{1}{3a^2 R^2} \left\{ y - \frac{3q}{2} + \frac{p}{2a^3} \left(x - \frac{p}{2a} \right) (y - q) \right. \\ &\quad \left. + \frac{q}{2a^2} \left[- \left(x - \frac{p}{2a} \right)^2 + (y - q)^2 \right] \right\}. \quad (12.9) \end{aligned}$$

Figures 14 – 18 show the results of integrating (5.1) –

(5.7) with P , Q and S given by (12.2) and (9.2), and the values of the parameters given by (12.4) and (10.1). The numerical procedure was the following:

1. The values of x and y at the maximum and at the minimum of the dipole were calculated from (12.5) for every r in the range $[0, 2]$ with the step $\Delta r = 1/300\,000$.

2. The initial values of r and t were the same as for ray 1b in Sec. X:

$$(r, t)_o = (0, 1.2). \quad (12.10)$$

3. The initial values of x and y on the rays were chosen by trial and error, so as to approach the direction in which strong blueshifts can be expected.

4. As explained in the paragraph below (12.7), the initial values of k^x and k^y were chosen zero.

5. For each pair (x_o, y_o) , the initial values of $(k^x/\Phi)_o$ and $(k^y/\Phi)_o$ were calculated from (12.8) and (12.9).

6. The initial value of k^t was taken -1 , in agreement with (5.8). The initial value of k^r was then calculated from (5.7), knowing that $k_o^x = k_o^y = 0$ and $k^r > 0$ (at the center $r = 0$ there is no other possibility for k^r).

7. The step in the affine parameter was chosen $\Delta\lambda = 10^{-7}$. The initial value of λ was irrelevant, since λ does not appear explicitly in any of the graphs.

8. Given the above, the next value of each function $f(\lambda)$ was calculated from

$$f(\lambda + \Delta\lambda) = f(\lambda) + \frac{df}{d\lambda} \Delta\lambda. \quad (12.11)$$

9. At each λ , $d(t, r, x, y)/d\lambda$ were calculated from (5.1), $d(k^t, k^x, k^y)/d\lambda$ were calculated from (5.3) and (5.5) - (5.6). With these data, the values of $(t, r, x, y, k^t, k^x, k^y)$ at $\lambda + \Delta\lambda$ were found using (12.11), and the value of k^r at $\lambda + \Delta\lambda$ was found from (5.7) assuming $k^r > 0$.

The values of x and y at the dipole maximum and minimum at $r = 0$ are

$$\begin{aligned} x_{\max}^o &= 0.36830403011403989, \\ y_{\max}^o &= 0.76587184263184405, \\ x_{\min}^o &= -5.61304662256229547 \times 10^{-2}, \\ y_{\min}^o &= 0.22899995223995073. \end{aligned} \quad (12.12)$$

Table IV gives the initial values of x and y for the various rays shown in the figures and the local minimum of $1+z$ achieved on each ray. Where the smallest $1+z=1$, the redshift along the ray was monotonically increasing.

The initial points of the rays come in 4 sets. The first set consists of rays A – F, which had initial y at y_{\max}^o , and initial x varied by trial and error around x_{\max}^o .

The second set consists of rays G – K, for which the initially chosen $x_1 - p/(2a)$ and $y_1 - q$ were multiplied by α to obtain

$$[x_2 - p/(2a), y_2 - q] = \alpha[x_1 - p/(2a), y_1 - q]. \quad (12.13)$$

This was meant to change $\sqrt{[x - p/(2a)]^2 + (y - q)^2}$ while keeping $(y - q)/[x - p/(2a)]$ unchanged (this meant

changing ϑ while keeping φ unchanged in the polar graphs in Figs. 15, 16 and 17). The variable parameters were x_1 and α , both chosen by trial and error. For ray G $\alpha = 0$, for the other rays in this group $\alpha = 0.06$. The parameter x_1 on rays H – K was βx_{\max}^o , with β being 0.2 on ray H, 0.552 on ray J and 0.554 on ray K.

The third set consists of rays L and M. They have minimum z closer to -1 than on the other rays. For ray L $\alpha = 0.09$ and $\beta = 0.65$; for ray M $\alpha = 0.088$ and $\beta = 0.6452$. Attempts at nailing down the expected SB ray with larger precision were becoming prohibitively time-consuming and were abandoned.

Rays A – M had their initial directions in a vicinity of the dipole maximum. The rays in the fourth set, N and O, had their initial directions in a vicinity of the dipole minimum. For ray N, $\alpha = 5$ and $x_1 = -1.33x_{\min}^o$; for ray O $\alpha = 0.06$ and $x_1 = 0.554x_{\min}^o$. Ray N was meant to demonstrate that a $z(r)$ profile indicating proximity to an SB ray exists also in a vicinity of the dipole minima. Attempts at further improvement were abandoned when they became too time-consuming.

Ray O is an example of a weird behaviour caused by proximity to an SB ray; see further below.

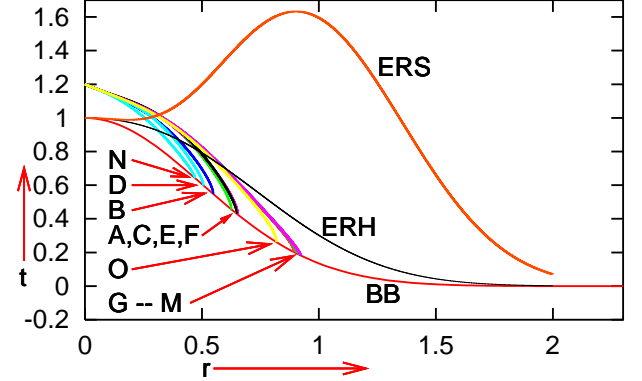


FIG. 14: The $t(r)$ graphs of rays A – O.

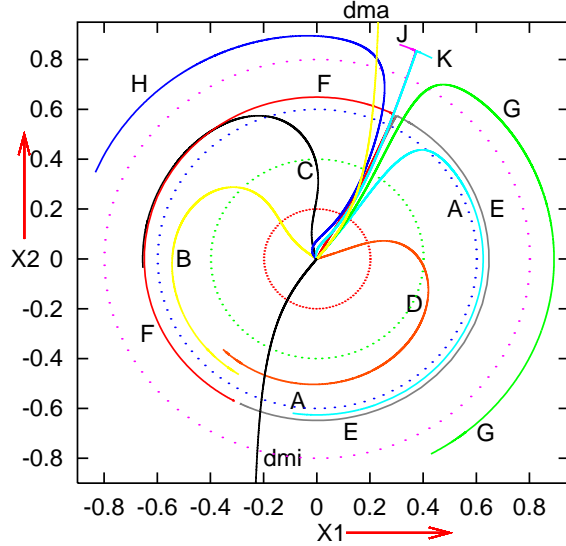
Figure 14 shows the $t(r)$ graphs of rays A – O. The ERS was calculated only for axial rays in the axially symmetric case, so it is an extraneous element in the general case. But it corresponds to the same BB profile, so is drawn for comparison. The ERH is the analogue of the ERS in the L–T limit, also calculated for the same BB profile.

Figures 15 and 16 show the $r(\varphi)$ graphs of all the rays, where $\varphi(r)$ and $\vartheta(r)$ are related to $x(r)$ and $y(r)$ by (2.3). The coordinates in these figures are $X_1 = r \cos \varphi$ and $X_2 = r \sin \varphi$, where $\tan \varphi = (y - Q)/(x - P)$. These are abstract graphs that are not projections of the rays on any actual subspace of the Szekeres manifold. They illustrate the process of approaching the expected SB paths.

The dotted circles are curves of constant r . The lines marked 'dma' and 'dmi' are the positions of the dipole maximum and minimum, respectively, for each r . Figure 15 shows the paths of the rays that have their initial points in the vicinity of 'dma' at $r = 0$; call them rays

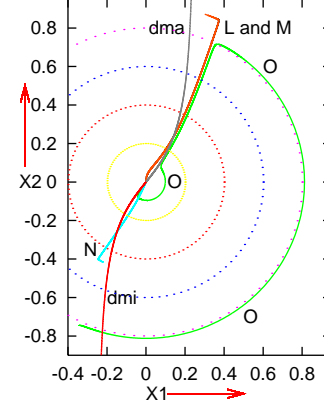
TABLE IV: Properties of the rays from Figs. 14 – 18.

Ray	Initial x	Initial y	min. $1+z$
A	x_{\max}^o	y_{\max}^o	1
B	0	y_{\max}^o	1
C	$x_{\max}^o/2$	y_{\max}^o	1.3576
D	$2x_{\max}^o$	y_{\max}^o	1
E	$0.88 \times x_{\max}^o$	y_{\max}^o	1.3274
F	$0.875 \times x_{\max}^o$	y_{\max}^o	1.3261
G	$p/(2a)$	q	0.511417
H	$0.47p/a + 0.012x_{\max}^o$	$0.94q + 0.06y_{\max}^o$	0.44733274
J	$0.47p/a + 0.03312x_{\max}^o$	$0.94q + 0.06y_{\max}^o$	0.319953
K	$0.47p/a + 0.03324x_{\max}^o$	$0.94q + 0.06y_{\max}^o$	0.3194256
L	$0.405p/a + 0.0585x_{\max}^o$	$0.81q + 0.09y_{\max}^o$	0.12313
M	$0.456p/a + 0.0567776x_{\max}^o$	$0.912q + 0.088y_{\max}^o$	0.06946378
N	$-2p/a - 6.51x_{\min}^o$	$-4q + 5y_{\min}^o$	0.19046
O	$-0.47p/a - 0.0324x_{\min}^o$	$0.94q + 0.06y_{\min}^o$	0.8589

FIG. 15: The $r(\varphi)$ graphs of rays A – K (in polar coordinates). See text for explanations.

of class 1. Curve A had the initial direction tangent to the 'dma' path and was the first trial. Rays B – K show consecutive approximations to the SB path obtained by moving the initial x and y by trial and error around x_{\max}^o and y_{\max}^o . The SB path should lie near to J and K.

Figure 16 shows four more rays; they are drawn separately to avoid clogging the image. Curves L and M are the paths of class 1 rays that hit the BB at still larger r than J and K, and are still better approximations to the expected SB path, with curve M being the best approximation; see Table IV and Fig. 18 further below. Curves N and O have their initial (x, y) in the vicinity of 'dmi' at $r = 0$, call them rays of class 2. Curve N is a good

FIG. 16: The $r(\varphi)$ graphs of rays L – O (in polar coordinates). Rays L and M nearly coincide, except that L goes farther. See text for explanations.

approximation to a second SB path.

The instability around the SB ray in class 2 is different than in class 1. In class 1, rays close to the SB path bend away from it, while in class 2 they bend toward it. We observed the same characteristic pattern in Fig. 10.

Curve O is an example of a strange path that results when the initial (x, y) in class 2 is close to $[p/(2a), q]$ and the initial direction is close to the 'dmi' path. This curve begins close to 'dmi', then bends away from it, goes around half a circle at nearly constant r , then proceeds near the opposite SB path, and finally, near the BB, bends again to once more go around half a circle at nearly constant r . The minimum redshift along it is moderately negative; see Table IV and Fig. 18.

Rays M and N which are expected to be near to the SB rays do meet the BB nearly orthogonally to the $r =$ constant surfaces, but are not related to the dipole max-

ima and minima, unlike in the axially symmetric case in Fig. 10. Thus, the coincidence between the SB rays and the dipole extrema in that case was forced by the symmetry of the model, and is not generic. This is still one more warning that by studying models with symmetries we may have forced independent objects to coincide.⁴

It follows that the SB rays determine preferred directions in general quasi-spherical Szekeres spacetimes that are independent of the trajectories of the dipole extrema.

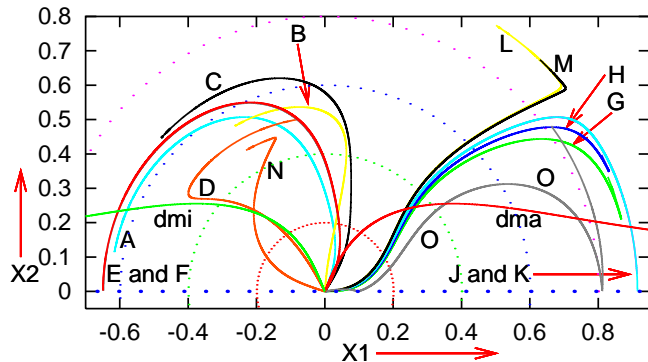


FIG. 17: The $r(\vartheta)$ graphs of rays A – O.

Figure 17 shows the $r(\vartheta)$ graphs of all the rays. Like Figs. 15 and 16, this is an abstract graph which is not supposed to represent a projection of the rays on any actual subspace. The horizontal dotted line is $\vartheta = 0$. By definition, ϑ must obey $0 \leq \vartheta \leq \pi$, so no curve can cross the line $\vartheta = 0$, and this is why curve O bounces off it close to the point $r = 0$. The sharp turns on curves G, H, L, M, N and O occur where they are near the BB and begin to approach it tangentially to a surface of constant r . In this graph, the curves that are supposed to be near to

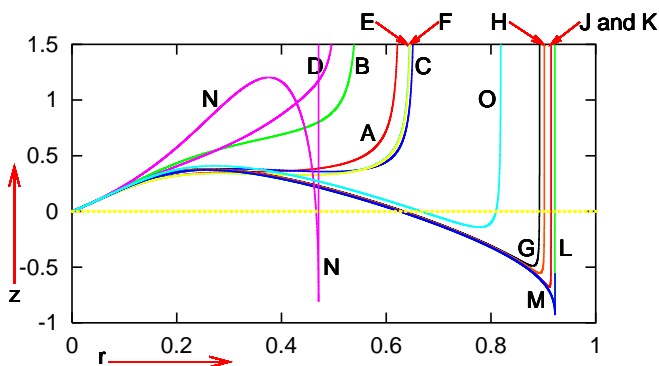


FIG. 18: The $z(r)$ redshift profiles for rays A – O. The dotted horizontal line marks $z = 0$.

the SB paths (L, M and N) do not approach the circular segment orthogonally to the $r = \text{constant}$ lines.

Finally, Fig. 18 shows the $z(r)$ profiles along all the rays. The profiles along rays G – M form a sequence similar to {II, I} in Fig. 9 that approached the 1b profile. At the scale of Fig. 18 M overlays L, but, as seen from Table IV, has a smaller minimum of redshift. The profile along ray N is similar to profile VIII; both 1b and VIII were shown to be the actual strongly blueshifted rays. Consequently, rays M and N must be near to SB rays.

XIII. CONCLUSIONS AND SUMMARY

After preliminary discussions and definitions in Secs. II – VIII, we introduced an exemplary axially symmetric quasi-spherical Szekeres (QSS) spacetime in Sec. IX. We then showed numerically in Sec. X that light rays exist in it that display strong blueshifts ($1 + z \approx 10^{-6}$) when they are emitted near the Big Bang (BB). The strongly blueshifted (SB) rays intersect every space of constant t on the axis of symmetry. In this case, the axes of the mass dipoles coincide with the axis of symmetry – and, automatically, also with the spatial paths of the SB rays.

In an L–T model, if a hump in the BB generates blueshift, then the blueshift is the same along rays passing through the center in all directions. In the axially symmetric Szekeres model, as the numerical computations show, the blueshift would be strong only along such rays whose directions at origin are tangent or near to the surfaces $u = 0$ and $u = +\infty$. Consequently, a present observer could see the blueshift only if she were placed near or in one of these surfaces.

The collection of rays emitted at the BB in an L–T model, on which the observed $z = -1$, is labelled by three parameters: two of them (the initial values of x and y) define the radial direction, the third one defines the radial coordinate r_e of the emission point on the BB where $t_e = t_B(r_e)$. In the axially symmetric model (7.4) – (7.5), as shown by the numerical experiments in Sec. X and XI, the collection of such rays is labelled by two parameters. One parameter is the value of φ (which is constant on each ray), the other parameter, the initial value of r , determines the emission point of the ray at the BB. So, the collection of all rays that have strong blueshifts is in this case a two-dimensional set.

Each SB ray is unstable: rays in every small neighbourhood of it all have infinite redshift between the BB and any later observer. Therefore, tracking the SB rays numerically is a rather hopeless undertaking: in the axially symmetric case we knew in advance where to expect them, and the condition of staying on the SB path was built into the numerical code. Nevertheless, such an attempt was undertaken in Sec. XII, in a QSS model without any symmetry. It was shown that sequences of rays exist in it, along which the redshift profiles are similar to those on rays that approach an SB ray in the axially symmetric case. The tentative conclusion is that SB rays

⁴ Well-known examples of this phenomenon are the stationary limit hypersurface in the Kerr metric [19] and the apparent horizon in the L–T models [20], which are in general distinct from the event horizon, but coincide with it in the Schwarzschild limit.

should exist also in general QSS models. The numerical calculations show clearly that in the nonsymmetric case the spatial paths of the SB rays are not related to the paths of the mass dipole extrema. Thus, the SB rays define other preferred directions in general QSS spacetimes.

Preferred null directions immediately evoke the association with principal null directions (PNDs) of the Weyl tensor. But in the nonsymmetric QSS model the SB rays do not coincide with the PNDs. The Weyl tensor of a general Szekeres spacetime is of Petrov type D, and in the coordinates of (2.1) its PNDs are

$$k_{\pm}^{\alpha} = \left(1, \pm \frac{\sqrt{1+2E}}{\Phi_{,r} - \Phi \mathcal{E}_{,r}/\mathcal{E}}, 0, 0\right). \quad (13.1)$$

These are geodesic only in the axially symmetric case; then they coincide with the $\{u = 0, k^{\varphi} = 0\}$ rays of Sec. X (they remain geodesic and SB also in the L–T limit). Consequently, in a general model the SB rays are independent of the PNDs (this is one more example to the phenomenon mentioned in footnote 4).

It remains a challenge to prove by exact methods that SB rays do exist in every QSS spacetime. Incidentally, such a proof is still lacking even in the L–T limit. It was proved exactly that rays emitted at the BB nonradially generate infinite redshifts [5], so rays emitted at the BB *can possibly* generate infinite blueshifts only if they are radial. But no exact proof exists that indeed $1+z = 0$ on those rays from the BB that do not prohibit it. All that is available is a casual remark without proof by Szekeres [3] and confirmations of it by perturbative calculations [4], and by numerical calculations in selected explicit examples of L–T spacetimes [5, 18].

The toy models considered in this paper were used to demonstrate the following three facts that should be useful in constructing realistic QSS models:

1. Rays with strong blueshifts do exist in an axially symmetric QSS model, and most probably also exist in QSS models without any symmetry.
2. The blueshift generated in these models is strongly anisotropic (it exists around only two directions).
3. The function S in (2.2) allows one to increase the size of the blueshift-generating region without changing the BB profile.

Appendix A: Solutions of (2.4) with $\Lambda = 0$

With $\Lambda = 0$, the solutions of (2.4) are the following: when $E(r) < 0$:

$$\begin{aligned} \Phi(t, r) &= -\frac{M}{2E}(1 - \cos \eta), \\ \eta - \sin \eta &= \frac{(-2E)^{3/2}}{M}[t - t_B(r)]; \end{aligned} \quad (A1)$$

when $E(r) = 0$:

$$\Phi(t, r) = \left\{ \frac{9}{2} M(r) [t - t_B(r)]^2 \right\}^{1/3}; \quad (A2)$$

when $E(r) > 0$:

$$\begin{aligned} \Phi(t, r) &= \frac{M}{2E}(\cosh \eta - 1), \\ \sinh \eta - \eta &= \frac{(2E)^{3/2}}{M}[t - t_B(r)]. \end{aligned} \quad (A3)$$

Throughout this paper only the case $E > 0$ is considered.

Appendix B: The Riemann tensor for (2.1)

In the orthonormal tetrad defined by the metric (2.1), the tetrad components of the Riemann tensor are:

$$R_{0101} = -\frac{M}{\Phi^3} + \frac{3M\Phi_{,r}}{\Phi^3\mathcal{N}} - \frac{M_{,r}}{\Phi^2\mathcal{N}}, \quad (B1)$$

$$R_{0202} = R_{0303} = \frac{1}{2}R_{2323} = -\frac{M}{\Phi^3}, \quad (B2)$$

$$R_{1212} = R_{1313} = -\frac{2M}{\Phi^3} + \frac{3M\Phi_{,r}}{\Phi^3\mathcal{N}} - \frac{M_{,r}}{\Phi^2\mathcal{N}}, \quad (B3)$$

where \mathcal{N} is given by (5.2). Note that at the origin, where $\Phi/\mathcal{E} = 0$, if $|\mathcal{E}_{,r}| < \infty$ we have $\mathcal{N} = \Phi_{,r}$, and the components of R_{ijkl} are finite as long as $|M_{,r}/\Phi_{,r}| < \infty$.

Appendix C: The general equation of an ERS

We keep the assumptions $u = 0$ and $\varphi = \text{constant}$ along a null geodesic, and take up the reasoning at (8.4). Using (4.2), Eq. (8.3) can be written as

$$\begin{aligned} &\left(\frac{E_{,r}}{2E} - \frac{\mathcal{E}_{,r}}{\mathcal{E}}\right) \frac{\Phi^2 \Phi_{,t}}{M} \\ &- \left(\frac{3}{2} \frac{E_{,r}}{E} - \frac{M_{,r}}{M}\right) (t - t_B) = -t_{B,r}. \end{aligned} \quad (C1)$$

Using (A3) in the above to eliminate Φ , $\Phi_{,t}$ and $M/(2E)^{3/2}$, one obtains

$$\begin{aligned} (t - t_B) &\left[\left(\frac{E_{,r}}{2E} - \frac{\mathcal{E}_{,r}}{\mathcal{E}}\right) F(\eta) + \frac{M_{,r}}{M} - \frac{3}{2} \frac{E_{,r}}{E} \right] \\ &= -t_{B,r}, \end{aligned} \quad (C2)$$

where

$$F(\eta) \stackrel{\text{def}}{=} \frac{(\cosh \eta - 1) \sinh \eta}{\sinh \eta - \eta}. \quad (C3)$$

The no-shell-crossing conditions imply $-t_{B,r} \geq 0$ [13]. We have $F(\eta) > 3$ and $dF/d\eta > 0$ for all $\eta > 0$ and $\lim_{\eta \rightarrow \infty} F(\eta) = \infty$. The coefficient of $F(\eta)$ in (C2) is positive in consequence of (2.8). The term $M_{,r}/M - (3/2)E_{,r}/E$ is not guaranteed to be positive, but is independent of η , so, with sufficiently large η the first term in the square brackets will dominate over it and will make the whole left-hand side of (C2) positive. Thus, with sufficiently large η (implying large $t - t_B$), the

left-hand side of (C2) will be larger than $(-t_{B,r})$, at any fixed r . On the other hand, at $t = t_B$ (i.e. $\eta = 0$), the left-hand side of (C2) is zero, so smaller than $(-t_{B,r})$. Consequently, at every r (C2) has a solution for t somewhere in the range $t_B < t < \infty$, and so an ERS exists.

Appendix D: A Friedmann limit of the Szekeres spacetime

Assume M and E to be of the Friedmannian forms (3.2), (8.5) and $t_B = \text{constant}$. Then (A1) – (A3) imply

$$\Phi(t, r) = rS(t); \quad (\text{D1})$$

eq. (A2) shows this immediately, in (A1) and (A3) one first has to note that η is independent of r .

Next, substituting (3.2) and (D1) in (2.5) we obtain

$$\kappa\rho = \frac{6M_0}{S^3} \quad (\text{D2})$$

because the factor $1 - r\mathcal{E}_{,r}/\mathcal{E}$ that appears in the numerator and denominator of (2.5) is nonzero and cancels out. Equation (D2) implies that ρ depends only on t in this limit. Thus, the set of equations $\{(3.2), (8.5), (\text{D1})\}$ is a sufficient condition for the density to become spatially homogeneous, i.e. for (2.1) to become the Friedmann metric, with no conditions on P , Q and S .

Appendix E: Proof of (12.1) with (12.3)

With P and Q given by (12.2), Eq. (12.1) becomes

$$\frac{1}{r^2} + \frac{1}{a^2 + r^2} - \frac{p^2 r^2}{(a^2 + r^2)^4} - \frac{q^2 r^2}{(a^2 + r^2)^3} > 0. \quad (\text{E1})$$

Denoting $r^2 = \chi$, this can be rewritten as

$$F(\chi) \stackrel{\text{def}}{=} (a^2 + \chi)^4 + \chi(a^2 + \chi)^3 - q^2 \chi^2(a^2 + \chi) - p^2 \chi^2 > 0. \quad (\text{E2})$$

This holds at all $\chi > 0$ if p and q obey (12.3).

Appendix F: Calculations underlying Figs. 11 – 13

Figure 19 illustrates the numerical calculations underlying Figs. 11 – 13. As with Fig. 1, the mapping into the Euclidean space was done so that the shortest distances between the spheres (measured along the dipole maximum) were preserved. For clarity, the figure shows the great circles of those spheres as if they all lied in the (X_1, X_2) plane of the Euclidean space, but the calculations were done in 3 dimensions.

The 3 spheres (call them S_1 , S_2 and S_3) have their centers at O_1 , O_2 , O_3 and their radii are, respectively,

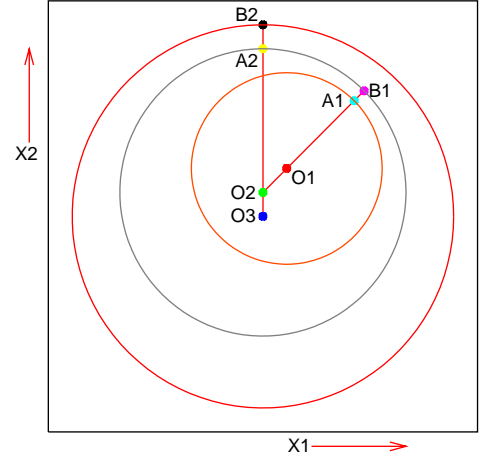


FIG. 19: Illustration to the calculations underlying Figs. 11 – 13. See text for explanations.

the distances $|O_1 A_1|$, $|O_2 B_1|$ and $|O_3 B_2|$. The figure illustrates how the calculation proceeds from sphere S_1 to S_2 and then from S_2 to S_3 .

The point A_1 is the position of the dipole maximum on sphere S_1 . Given the r -coordinate of A_1 (from the previous step of calculation), its (x, y) coordinates are calculated from (12.5) and then converted to the (ϑ, φ) coordinates by (2.3). The (ϑ, φ) coordinates are assumed to coincide with the spherical polar coordinates in the Euclidean space of Figs. 11 – 13. (These coordinates are defined even in the limit $r \rightarrow 0$ because (12.5) has a well-defined limit $r \rightarrow 0$.)

The radius $|O_1 A_1|$ is calculated using (9.5). With $P(r)$ and $Q(r)$ being nonconstant, the path of $\mathcal{E}_{,r} = 0$ is no longer straight, but (9.7) still holds, so (9.5) still makes sense as the definition of the geodesic radius. Then, $|A_1 B_1|$ is calculated from (9.10) using (12.6) with $\varepsilon = +1$ for the value of $\mathcal{E}_{,r}/\mathcal{E}$. At the point B_1 , the r -coordinate of the sphere S_2 is known, so we calculate its geodesic radius from (9.5); this radius is $|O_2 B_1|$.

Given the Euclidean (X_1, X_2, X_3) coordinates of O_1 (also from earlier calculation) and the distances $|O_1 A_1|$, $|A_1 B_1|$ and $|O_2 B_1|$ we calculate the position of the center of O_2 by adding the quantity $(|O_1 A_1| + |A_1 B_1| - |O_2 B_1|)$, projected on the (X_1, X_2, X_3) Euclidean axes, to the coordinates of O_1 . Having thus determined the (X_1, X_2, X_3) coordinates of O_2 , we calculate the (ϑ, φ) coordinates of A_2 (the dipole maximum on sphere S_2) from (12.5) and (12.3), and proceed by the same method to determine the geodesic radii and the center O_3 of the next sphere S_3 .

The calculation begins by assuming a position for the sphere of radius 0 (in Figs. 11 – 13 it is $X_1 = X_2 = X_3 = 0$). The centers and radii of all the larger spheres are then calculated as described above.

This calculation involves an approximation, on top of the obvious approximation connected with the numerical calculations. It assumes that the segments $A_1 B_1$ and

$A_2 B_2$ are straight, so that $\mathcal{E}_{,r}/\mathcal{E}$ along them depends only on r and is given by (12.6).

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- [1] G. Lemaître, L'Univers en expansion [The expanding Universe]. *Ann. Soc. Sci. Bruxelles* **A53**, 51 (1933); English translation: *Gen. Relativ. Gravit.* **29**, 641 (1997); with an editorial note by A. Krasinski: *Gen. Relativ. Gravit.* **29**, 637 (1997).
 - [2] R. C. Tolman, Effect of inhomogeneity on cosmological models. *Proc. Nat. Acad. Sci. USA* **20**, 169 (1934); reprinted: *Gen. Relativ. Gravit.* **29**, 935 (1997); with an editorial note by A. Krasinski, in: *Gen. Relativ. Gravit.* **29**, 931 (1997).
 - [3] P. Szekeres, Naked singularities. In: *Gravitational Radiation, Collapsed Objects and Exact Solutions*. Edited by C. Edwards. Springer (Lecture Notes in Physics, vol. 124), New York, pp. 477 – 487 (1980).
 - [4] C. Hellaby and K. Lake, The redshift structure of the Big Bang in inhomogeneous cosmological models. I. Spherical dust solutions. *Astrophys. J.* **282**, 1 (1984) + erratum *Astrophys. J.* **294**, 702 (1985).
 - [5] A. Krasinski, Cosmological blueshifting may explain the gamma ray bursts. *Phys. Rev.* **D93**, 043525 (2016).
 - [6] P. Szekeres, A class of inhomogeneous cosmological models. *Commun. Math. Phys.* **41**, 55 (1975).
 - [7] P. Szekeres, Quasispherical gravitational collapse. *Phys. Rev.* **D12**, 2941 (1975).
 - [8] J. Plebański and A. Krasinski, *An Introduction to General Relativity and Cosmology*. Cambridge University Press 2006, 534 pp, ISBN 0-521-85623-X.
 - [9] C. Hellaby and A. Krasinski, Physical and Geometrical Interpretation of the $\epsilon \leq 0$ Szekeres Models. *Phys. Rev.* **D77**, 023529 (2008).
 - [10] A. Krasinski, Geometry and topology of the quasiplane Szekeres model. *Phys. Rev.* **D78**, 064038 (2008) + erratum *Phys. Rev.* **D85**, 069903(E) (2012). Fully corrected text: arXiv:0805.0529v4.
 - [11] A. Krasinski and K. Bolejko, Geometry of the quasihyperbolic Szekeres models. *Phys. Rev.* **D86**, 104036 (2012).
 - [12] C. Hellaby, The nonsimultaneous nature of the Schwarzschild $R = 0$ singularity. *J. Math. Phys.* **37**, 2892 (1996).
 - [13] C. Hellaby and A. Krasinski. You cannot get through Szekeres wormholes or regularity, topology and causality in quasi-spherical Szekeres models. *Phys. Rev.* **D66**, 084011 (2002).
 - [14] M. M. de Souza, Hidden symmetries of Szekeres quasi-spherical solutions, *Revista Brasileira de Física* **15**, 379 (1985).
 - [15] K. Bolejko, A. Krasinski, C. Hellaby and M.-N. Célérier, Structures in the Universe by exact methods – formation, evolution, interactions. Cambridge University Press 2010, 242 pp, ISBN 978-0-521-76914-3.
 - [16] G. F. R. Ellis, Relativistic cosmology. In *Proceedings of the International School of Physics “Enrico Fermi”, Course 47: General relativity and cosmology*. Edited by R.K. Sachs, Academic Press, 1971, pp. 104-182. Reprinted: *Gen. Relativ. Gravit.* **41**, 581 (2009); with an editorial note by W. Stoeger, in *Gen. Relativ. Gravit.* **41**, 575 (2009).
 - [17] B. C. Nolan and U. Debnath, Is the shell-focusing singularity of Szekeres space-time visible? *Phys. Rev.* **D76**, 104046 (2007).
 - [18] A. Krasinski, Blueshifts in the Lemaître – Tolman models. *Phys. Rev.* **D90**, 103525 (2014).
 - [19] B. Carter, Black hole equilibrium states. Part I: Analytic and geometric properties of the Kerr solutions, in: *Black Holes – les astres occlus*. Edited by C. de Witt and B. S. de Witt. Gordon and Breach, New York, London, Paris, p. 61 (1973). Revised reprinting: *Gen. Relativ. Gravit.* **41**, 2874 (2009); with an editorial note by N. Kamran and A. Krasinski, *Gen. Relativ. Gravit.* **41**, 2867 (2009).
 - [20] A. Krasinski and Charles Hellaby, Formation of a galaxy with a central black hole in the Lemaître – Tolman model. *Phys. Rev.* **D69**, 043502 (2004).